Estimating the correlation of bivariate normally distributed destructive strength properties: breaking the same board twice-again.*

Yanling Cai¹, , Will Welch¹ and James V. Zidek¹ and Conroy Lum²

¹ Department of Statistics, University of British Columbia
² FPInnovations, Vancouver, Canada

Abstract

The paper shows how to find the maximum likelihood estimate (MLE) of the correlation between a pair of normally distributed responses \(X\) and \(Y\). These represent destructively measured strengths so only one can be observed on each specimen. The method involves first stressing the specimen on say the \(X\)-mode up to a level \(l_1\) and if it does not fail, stressing the same specimen on the \(Y\)-mode until it does. Since the preliminary test will damage any surviving specimen, not \(Y\) but instead \(Y^*\) is observed so the computation of the MLE is not straightforward. The paper shows how to compensate for that damage and simulation studies demonstrate the success of the method. Use of the method is demonstrated with an application to a large-scale destructive testing experiment on manufactured lumber specimens that were carried out by the authors’ research group. There the variables were the logged tensile strength and logged modulus of rupture.

Keywords: size censored measurements; penalized maximum likelihood; power prior; lumber

*The work reported in this paper was supported by a Collaborative Research and Development grant from the Natural Science and Engineering Research Council of Canada as well as FPInnovations in the form of in-kind contributions as well as guidance.
strength; reliability; competing risks
1. INTRODUCTION

This paper is about estimating the correlation $\rho$ between normally distributed random variables $X$ and $Y$. Were independent measurements of the vector pair $(X, Y)$ available on each specimen in the trial, this would be straightforward. However in situations addressed in this paper only one of the two responses can be measured on each piece. More precisely only a response $Z$ is observed where $Z = X$ if $X \leq l_1$ by destructively testing in the $X$-mode up to a level $l_1$, a known threshold set by the experimenter, and the specimen fails. But if the specimen survives, i.e. $X > l_1$ then the survivor is tested in the $Y$-mode until failure in which case the observation is $Z = Y^* < Y$ since the initial test will damage the specimen.

Our results pertain to two broad paradigms of statistical experimentation seen in practice. In the first $X$ is censored by size and not observed if $X \geq l_1$. Such a situation arises in accelerated life testing of products that may be expected to operate for thousands of hours without failure under normal conditions. But to conduct preliminary tests of the product at those levels would lead to unacceptably long delays in getting the product into a competitive market. Thus commonly industry will test the product at higher than expected loads in the tests to accelerate the time to failure. However that then requires knowledge not always available, of the relationship between the accelerated failure time and the normal failure time $X$. When that knowledge is not available, an alternative solution might be used, namely find a second variable $Y$ correlated with $X$ that fails in shorter time. The test would then consist of stressing the product at normal levels to a specified time or load $l_1$ and observing $X$ if it fails and $Y^*$ if it does not. The result will be $Z$ and the need to estimate $\rho$ in order to predict the unobserved $X$ from $Y^*$.

The second paradigm, is the one seen in the application described in this paper. Here both $X$ and $Y$ are readily observable but the cost of collecting two samples for the purpose can be expensive say in a long term monitoring or quality control setting. The cost can be prohibitively expensive when a large number of characteristics need to be monitored, but no two can be measured on the same specimen.

Now observing $Z$ described above provides a information about the relationship between $X$ and $Y$ so that in future, one could be predicted from the other. But the amount of information depends on the relationship between $Y$ and $Y^*$. This paper is about determining that relationship and how
the nonnormally distributed $Y^*$ can be used to help estimate the correlation $\rho$ between $X$ and $Y$.

Its genesis lies in the long term monitoring program for the strength of manufactured lumber, which is now being established in North America. Concern here lies in the changing mix of lumber species as well as in the possible effects on lumber strength of climate changes – trees are growing differently. The ultimate concern lies in structural engineering and the use of this sustainable building material. The estimated $\rho$ provides a predictor of $X$ given $Y$.

The idea of using $Z$ in this way when $Y^* = Y$ seems to be due to (Evans, Johnson and Green 1984), who referred to the process as “breaking the same board twice”. They proposed a single proof load design (SPLD) for that purpose. Proof loading is a standard technique used in industry to pre–tests products to a pre-determined, usually low load level. Survivors which are assumed to undamaged given the low load level, are passed on to the consumer. In the next section, we will see that this idea can be generalized and applied to advantage in the second paradigm above, where observing $Z$ based on a proof-loading various levels $l_1$ yields an estimator of $\rho$, once the effects of damage are taken into account. (Our analysis would suggest an optimum level around the median of the $X$ distribution.)

However, the idea of Evans et al. (1984), although ingenious cannot be used in its original form. True, the classical theory for the maximum likelihood estimator of $\rho$ holds - asymptotic consistency and normality of the maximum likelihood estimator (MLE) $\hat{\theta}$ obtained from SPLD holds under an assumed bivariate normal distribution or 2-parameter Weibull distribution (Amorim and Johnson (1986) and Johnson and Lu (2007), respectively). However for samples of moderate size, the MLE for $\rho$ proves very unstable when $\rho$ is small.

It could be argued that the case of a small $\rho$ is unimportant—one would not want to predict $X$ given $Y$ in this case. But in many cases the magnitude of $\rho$ will not be known, a priori and the instability of the MLE of $\rho$ can lead to estimates that are quite large e.g. $\hat{\rho} = 0.5$ when in fact $\rho = 0.2$. Basing a predictor on this estimate would have potentially serious adverse consequences for example in a long term monitoring program.

Amorim and Johnson (1986) recognize this problem and propose a symmetric proof load design again where $Y^* = Y$, to assess the dependence between the destructive random variables. In this symmetric design, a sample of specimens is divided into two halves. One half is tested with a single
proof load design, with the initial proof loading being in one variable. The other half is done in a
similar way, but this time initial proof loading in another random variable.

But this solution is not always practical due to cost or other issues. In the context of the lumber
monitoring considered in Section 4 for example, proof loading is inappropriate for compression
strength properties leading to the proposal in this paper.

(Cai 2015a) rediscovers and alternate approach when \( Y = Y^* \) first proposed in Amorim and
Johnson (1986) where it is called a “hybrid” approach. Here the the specimens in a random sample
are assigned to one of the two groups: the SPLD group and the shoulder group of \( Y \). The specimens
in a “shoulder group” are all tested to failure in the \( Y \)-mode. This design we will refer to in this
paper as the “SPLD–with–shoulder”. Although this design inflates the cost of the experiment, it
reduces the instability in the estimator of \( \rho \).

In this paper we will assume that the SPLD–with–shoulder is being used. How to regularize the
unstable estimator of \( \rho \) without the shoulder remains an open problem, although a not altogether
satisfactory penalized MLE is discussed in (Cai 2015a).

But what is \( Y^* \neq Y \)? That is the main subject of this paper and that question so far as we
know, has not been previously been addressed in the degradation literature.

The paper will, in its applications focus on two important measures of strength that will play
the roles of \( X \) and \( Y \). They are respectively the modulus of rupture (MOR) and the ultimate tensile
strength (UTS) found by bending and stretching a piece of lumber until it fails. The paper is laid
out as follows. Section 2 develops a damage model after assessing some earlier empirical work for
completeness.

Although our methodology of assessing the dependence between the two random variables when
one of margins is censored by size is very general, we provide an illustration on the lumber applica-
tion. Then in Section ??

2. DAMAGE MODEL FOR PROOF LOADED LUMBER

The accumulation of strength damage in structural members due to both live as well as dead loads
has been extensively investigated in the past, in particular the duration of load effect. Early work
was largely empirical but later theoretical approaches were taken to enable scenario analysis for
long term load effects, where empirical approaches were not possible. Theoretical approaches have recently been reviewed in (Hoffmeyer and Sørensen 2007) and (Zhai 2011). For completeness, this section provides a brief account of the empirical as well as theoretical history.

2.1 Empirical studies

We start with some of the empirical studies of the damage caused by proof-loading. (Strickler, Pellerin and Talbott 1969) describes an experiment in proof loading structural end-jointed lumber in the MOR mode to investigate whether significant damage accumulates in the joints. Comparing the MOR averages of the proof loaded groups and the control group where specimens are tested to failure in the MOR mode without proof loading reveals no significant difference between the group averages. The paper finds no reduction in lumber strength due to proof loading even with loads up to 70 percent of the mean of the MOR distribution.

(Madsen 1976) describes an experiment conducted with a sample size exceeding 1000 of 2 by 6 inch specimens of length 12 feet selected from the “# 2 and better” grade of the Hem-Fir species group. Of specific interest was a destructive strength property called the “ultimate tensile strength (UTS)”, which is measured by applying a load parallel to the grain of the specimen. The specimens were randomly divided into 6 groups: two proof loaded groups that were tested to failure in the UTS mode in the same direction as the applied proof load (first category); two proof loaded groups that were tested to failure in the UTS mode in the opposite direction to the applied proof load (second category); one control group that tests specimens’ strong sides to failure in the UTS mode; one control group that tests specimens’ weak sides to failure in the UTS mode. The proof load stress was set at 4000 psi, leading to failures of between 5% to 10%, depending on the proof loaded group.

The paper makes a visual comparison of the empirical cumulative distributions of the proof loaded group and the control group for each category. For the first, a few of the surviving specimens failed below the proof load level during the follow up test, indicating damage. However, the empirical cumulative distribution of the proof loaded group generally tracked the control group closely and no apparent reduction in strength was observed as a result of proof loading. For the second category, many specimens failed below the proof load level. Comparing the distributions of the proof loaded group and the control group indicated that some damage may have occurred during the proof loading procedure for the second category. However, the differences observed in the distributions could also
be due to the test orientation or a mixed effect of damage and orientation.

(Marin and Woeste 1981) describes a reverse proof loading experiment with two hundred specimens in each of the control and proof loaded groups, with specimens proof loaded in the MOR mode. The proof load level is the 5th percentile of the MOR distribution. The authors of the paper fitted Weibull distributions to the MOR measurements of the proof loaded group and the MOR measurements of the control group, and compared the two fitted distributions. The proof loaded group has a right strength tail that resembles the right strength tail of the control group, but it has a much higher fifth percentile. Only one proof loaded specimen failed below the proof load level. Any damage which did not result in failure below the proof load level is not detected.

(Woeste, Green, Tarbell and Marin 1986) presents the results of an investigation of various proof loading procedures for assessing lumber strength. The specimens were assigned to six groups: three bending groups and three tension groups. Specimens in one of the three bending groups, the bending control group, were tested to failure in the MOR mode using random edge placement. Specimens in the second bending group, a single proof load sample, were proof loaded in the MOR mode using random edge placement. The survivors of the single proof load sample were then tested to failure in the MOR mode using the opposite edge. Specimens in the third bending group, a reverse proof load sample, were proof loaded in the MOR mode using both edges. The survivors of the reverse proof load bending group were tested to failure in the MOR mode. The three tension groups were designed in a similar way to the bending groups, but in the UTS mode. The percentages of specimens that failed during the proof loading procedures ranged from 7.1% to 11.1%. All survivors subjected to reverse bending proof loading failed above the proof load level. Seven out of 197 specimens subjected to a single bending proof load and loaded to failure on the opposite edge failed below the proof load level. This does not necessarily imply that the specimens were damaged due to proof loading. It could be due to the fact that the bending strength for one edge is not identical to the bending strength for the opposite orientation. The authors conclude that there was no apparent damage due to proof loading for both MOR or UTS.

(Katzengruber, Jeitler, Brandner and Schickhofer 2006) presents the results of an experiment with 4886 specimens proof loaded in the UTS mode to assure the quality of finger-jointed structural timber. The specimens were proof loaded to a first proof load level. Then after a short relief period,
the specimens were proof loaded again to a proof load level that was slightly higher than the first
proof load level. In total, 65 specimens failed because of the double stress proof loading procedure.
Among them, 37 specimens failed during the first proof loading procedure and 28 specimens failed
at the second proof load step. Only two specimens among the 28 specimens that failed at the second
proof load step failed at a level that was below the first proof load level. The authors concluded
that “within a double proof load procedure, 99.96% of all specimens could sustain higher stresses
than at first time, indicating not being damaged” \((2/4886 = 99.96\%)\).

In summary, the proof loading experiments found in the literature focus on studying damage
accumulated in the lumber strength when the specimens are proof loaded in one mode and the
survivors are tested to failure in the same mode. Proof load damage is assessed by comparing the
mean strength of the proof loaded group and the control group, by visual comparisons between
the lower quantiles of the strength distributions of the proof loaded groups and the control group
without any proof loading, or by testing the survivors to failure and checking whether they fail
below the proof load level (if so, indicating damage). Their results suggest that damage is negligible
when the proof load level is low (less than the 10th quantile of the strength distribution).

2.2 Theoretical damage accumulation models

One form of accumulated strength damage is called duration–of–load (DOL) effect when the lumber
specimens are exposed to both live and dead loadings over long periods of time. Unlike the previous
proof loading experiments, here lab testing and empirical assessment was not an option to quantify
duration–of–load effect and so instead the accumulated damage was modelling theoretically through
the use of dynamic models and differential equations. Analytical derivations of the accumulation
models then yield the desired relationships between the strength and the residual strength of a forest
product under stress, including in a special case that due to proof loading at low proof load levels.

The DOL effect implicates that the failure strength of lumber depends on how the load is
applied over time, i.e. on the loading profile \(\tau(t), \ t \geq 0\). One such profile, that defines the so–
called ramp–load test, increases the applied load at a constant rate per unit time \(\tau(t) = kt\) until
failure at a stress load of \(\tau_s\). Another such profile is that of the constant load test \(\tau(t) \equiv k\). For
practical reasons, a standard short–term strength parameter has been defined for each population
of wood products, and this is then downscaled to get the reduced failure strength due to the DOL
effect. In another application, the subject of this section, damage accumulation models for the
DOL effect have been developed that incorporate the short–term strength as a latent representative
of a specimen’s strength. Through the link in the model, this short–term strength “predicts” the
long–term strength that would obtain from a constant–loading test and hence the DOL effect. The
accumulated damage model thus exploits the fact that wood strength is a function of the time span
over which a load is imposed.

This section reviews three of the best known wood damage accumulation models: the Madison
curve (Wood 1951; Hendrickson, Ellingwood and Murphy 1987), the U.S. model (Gerhards and
Link 1986; Gerhards 1979) and the Canadian model (Foschi and Yao 1986). (Gerhards 1979)
suggests that the theoretical effect of proof loading on wood strength and lifetime could be predicted
from the damage accumulation models. After all, the same properties of the material govern the
underlying mechanism of the proof load effect and the DOL effect. For these reasons and for
completeness, we discuss how the residual strength of a specimen after proof loading depends on its
strength based on the wood scientists’ damage accumulation models.

The is characterized by a stochastic differential equations:

\[ \frac{d\alpha(t)}{dt} = g(\alpha(t), \sigma(t), \nu), \]  

where \( \alpha(t) \) is the damage accumulated by time \( t \), \( g \) is a function to be specified, \( \nu \) is a vector of
parameters, and \( \sigma(t) \) is the applied load ratio at time \( t \) as defined below. The accumulated damage
\( \alpha(t) \) is a non-decreasing function of \( t \), which takes any value between 0 and 1. Note that at time 0,
\( \alpha(0) = 0 \), indicating no damage has yet occurred. At a random time \( t = T \), a test specimen will fail
at which point the accumulated damage will be \( \alpha(T) = 1 \). The applied load ratio \( \sigma(t) = \tau(t)/\tau_s \) is
a dimensionless quantity.

(Cai 2015a) presents an extensive analysis of this dynamic equation and its special cases, with the
aim of obtaining an explicit model for accumulated damage. However the work does not successfully
yield a model that successfully captures an important qualitative feature of damage–for a survivor
of the proof loading first stage in \( X \)-mode \( Y^* \approx Y \) if \( X \) is large. Thus we omit the details for
brevity and refer instead to (Cai 2015a) and (Cai and Zidek 2016).
Summary Theoretical DOL damage accumulation models quantify the damage accumulated in the strength when the load is imposed in the same strength mode over time. However, interest in this paper lies in the damage accumulated in one strength when the load is imposed on lumber specimens in another strength mode. The DOL models do not seem helpful for characterizing this kind of damage, which leads to our empirical approach in the sequel.

2.3 A statistical approach

This section moves from the approaches described above that tend to focus on the effect of proof loading on individual specimens (albeit with specimen specific, random parameters), to one that focuses on the change in the population of a proof loading lumber. As a purely conceptual framework, imagine a wooden structure in a region that sustains a catastrophic terrestrial event such as an earthquake or hurricane. Even if its structural elements survive the live load imposed on them, they will be damaged, thus in effect moving them from their original parent population to another with weaker elements. From the perspective of reliability, the new population has a shorter return period. The novel approach described in this section provides a way of making that assessment. In fact the method predicts the effects induced by a live loading of one strength property on the return periods for the other strength properties of the survivor’s new populations.

We consider a randomly selected lumber specimen \(i\) with two strength properties \(X_i\) and \(Y_i\) as realizations of a random pair \(X\) and \(Y\), only one of which can be measured. We assume that \(X\) and \(Y\) have a bivariate normal distribution,

\[
\left( \begin{array} {c} X \\ Y \end{array} \right) \sim N_2 \left( \begin{array} {c} \mu_X \\ \mu_Y \end{array} \right), \begin{bmatrix} \sigma_X^2 & \rho \sigma_X \sigma_Y \\ \rho \sigma_X \sigma_Y & \sigma_Y^2 \end{bmatrix} \right).
\]

Let \(\theta = (\mu_X, \sigma_X, \mu_Y, \sigma_Y, \rho)^T\). The correlation between \(X\) and \(Y\) is \(\rho\).

In the SPLD experiment, the \(i\)th specimen is first proof loaded in the \(X\)-mode up to a pre-determined load level \(l\). If \(X_i \leq l\), the specimen fails and \(X_i = x_i\) is observed. But if \(X_i > l\), then the specimen is a "survivor" and \(X_i\) is not observed. The survivor is then tested to failure in the \(Y\)-mode. If the specimen does not fail during the proof loading procedure and the proof loading procedure does not damage the specimen, \(Y_i = y_i\) would then be observed. On the other hand,
damage may occur if \( l \) were set at a moderate to high level. In that case we observe \( Y_i^* = y_i^* \) instead of \( Y_i = y_i \) where \( Y_i^* \) is the \( i \)th specimen’s residual strength in the \( Y \)-mode after proof loading in the \( X \)-mode.

If we had also observed the \( \{Y_i\} \) for each survivor, it would not be difficult to assess the degree of the damage accumulated in the strength of the survivors due to proof loading. In this case, a “reference set” of \( Y \)-values is readily assembled for comparison with the \( Y^* \)-values. This leads us naturally then to the problem of determining the appropriate reference set, and thus the solution to our problem.

The damage assessment methods reviewed earlier are not applicable for the SPLD situation, as they would suggest comparing the distribution of \( Y^* \) conditional on \( X > l \) with the marginal distribution of \( Y \). This suggestion would not work in general. In fact the appropriate reference set depends on the unknown correlation \( \rho \) in the bivariate normal distribution. But the suggestion does work in the case \( \rho = 0 \). When \( \rho = 0 \), the size of \( Y \) would bear no relation to the size of \( X \). In particular, the unobserved \( Y_i \) corresponding to the \( Y_i^* \) would just be a random sample from the marginal \( Y \)-distribution as suggested above. At the other extreme when \( \rho = 1 \) (approximately), then apart from a linear transformation, \( X \) and \( Y \) would be about equal. In other words, the unobserved \( Y_i \) of the survivors would be like the unobserved \( X_i \), which are the largest values in the sample of specimens. If, for example, 40% of the specimens survived, we could regard these \( Y \)'s (the correct reference set) as having been sampled from the truncated \( Y \) distribution consisting only of values greater than the 60th percentile. The correct reference set at this extreme case when \( \rho = 1 \) would be the largest 40% of the values in a second group of \( Y \) values collected by testing a sizeable independent sample of specimens to failure in the \( Y \)-mode. If there were no damage accumulated in the survivors, an empirical Q-Q plot of \( Y^* \) against \( Y \) would lie close to the 45-degree line.

The question is what can be done in a more realistic case where \( \rho \) is unknown and \( 0 < \rho < 1 \)? The answer is presented in Section 2.3, which is equivalent to separate the effect of property dependence \( \rho \) and the effect of damage. We first estimate \( \rho \) and then as an approximation, treat it as known. We do this by running a second SPLD experiment with a proof load level set so low that little or no damage will occur in the survivors. Using the theory developed in the single proof load design with a shoulder approach, we may now estimate the parameters of the bivariate normal distribution.
of \(X\) and \(Y\), in particular the correlation parameter \(\rho\). Once these parameters are estimated, we
treat them as known. The known parameters in the bivariate normal distribution enable us to find
the estimated conditional distribution of \([Y|X > l]\) for any proof load level \(l\) even when \(l\) is large
enough that survivors could be damaged.

To assess the damage, we can compare the estimated theoretical quantiles of \([Y|X > l]\) with
the empirical quantiles of \([Y^*|X > l]\). The difference between the estimated theoretical quantile
of \([Y|X > l]\) and the empirical quantile of \([Y^*|X > l]\) provides information about the damage
if any damage has been sustained. In particular, if the estimated theoretical quantile is similar
to the empirical quantile, then no damage has occurred as a result of proof loading. The most
important aspect of this new approach is that it moves away from thinking about changes in
individual specimens to thinking about changes in the subpopulation of all specimens that are
stressed by a proof loading procedure. The approach considered in 2.3 also has other potential
applications. For example, it could be applied to relate strength properties measured by short–term
and long–term tests as in predicting the duration–of–load effect in a given population of lumber.

After introducing our method of estimating the damage accumulated in the strength properties
caused by proof loading in Section 2.3, simulation studies in Section 2.3 demonstrate its estimation
accuracy. The details now follow.

**Method description** The SPLD experiment enables us to observe random variables \(Z = XI\{X \leq l\} + YI\{X > l\}\) and \(U = I\{X \leq l\}\) where \(I\) denotes the (0 or 1) indicator function. The component
of the likelihood function of \(\theta\) for a single observation \(s = (z, u)\) is

\[
L(\theta; s) = \left\{f_X(x)\right\}^u \left\{f_{Y,X > l}(y, x > l)\right\}^{1-u} \\
= \left\{\exp\left[\frac{-(x-\mu_X)^2}{2\sigma_X^2}\right] \sigma_X \right\}^u \left\{f_Y(y) \int_{l}^{\infty} f_{X|Y}(x|y)dx\right\}^{1-u} \\
= \left\{\exp\left[\frac{-(x-\mu_X)^2}{2\sigma_X^2}\right] \sigma_X \right\}^u \left\{\exp\left[\frac{-(y-\mu_Y)^2}{2\sigma_Y^2}\right] \int_{a_l}^{\infty} e^{-a^2/2} \right\}^{1-u}
\]

(2)

where \(a_l = \{l - \mu_X - \rho(\sigma_X/\sigma_Y)(y - \mu_Y)\}/\{\sigma_X \sqrt{1 - \rho^2}\}\), \(f_X(x)\) is the density function of \(X\),
\(f_Y(y)\) is the density function of \(Y\), and \(f_{X|Y}(x|y)\) is the density function of \(X\) given \(Y\).
The component of the joint likelihood function of $\theta$ given the single observation $s = (z, u)$ from the SPLD group and the single observation $y$ from the shoulder group of $Y$ is

$$L_{PFY}(\theta; s, y) = L(\theta; s)L_Y(\mu_Y, \sigma_Y; y),$$

(3)

where $L_Y(\mu_Y, \sigma_Y; y) = \exp\{-(y - \mu_Y)^2/(2\sigma_Y^2)\}/(\sigma_Y\sqrt{2\pi})$ and $L(\theta; s)$ is Equation (2). Given a sample of observations $s = \{(z_1, u_1), (z_2, u_2), \ldots\}$ collected from the SPLD experiment and a sample of observations $y = \{y_1, y_2, \ldots, y_n\}$ collected from the shoulder group of $Y$, the likelihood function of $\theta$ is

$$L_{PFY}(\theta; s, y) = \left\{\prod_{i=1}^n L(\theta; s_i)\right\} \left\{\prod_{i=1}^{n_y} L_Y(\mu_Y, \sigma_Y; y_i)\right\}.$$

The parameter $\theta$ is estimated by the maximum likelihood method where the MLE maximizes the joint likelihood function $L_{PFY}(\theta; s, y)$. This approach is the SPLD–with–shoulder approach.

Given $Y = y$, the conditional random variable $[X|Y]$ follows a normal distribution with mean $\mu_X + \rho\sigma_X(y - \mu_Y)/\sigma_Y$ and variance $(1 - \rho^2)\sigma_X^2$. The density function of the conditional variable $[X|Y]$ is

$$f_{[X|Y]}(x|y) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_X} \exp\left\{-\frac{(x - \mu_X - \sigma_X\rho(y - \mu_Y)/\sigma_Y)^2}{2\sigma_X^2(1 - \rho^2)}\right\}.$$

(4)

In making the calculation in Equation (4) below, we have used this property of conditional normal distribution of a bivariate normal random variable.

The density function of $[Y|X > l]$ with $\theta = (\mu_X, \sigma_X, \mu_Y, \sigma_Y, \rho)^T$ is

$$f_{[Y|X > l]}(y; \theta) = \frac{f_{Y,X > l}(y, X > l; \theta)}{\mathbb{P}(X > l; \mu_X, \sigma_X)}$$

$$= \frac{1}{\mathbb{P}(X > l; \mu_X, \sigma_X)} \int_{l}^{\infty} f_Y(y; \mu_Y, \sigma_Y) f_{[X|Y]}(x|y; \theta) dx$$

$$= \frac{\exp\{-(y - \mu_Y)^2/(2\sigma_Y^2)\}}{\sigma_Y\sqrt{2\pi}(1 - \Phi([l - \mu_X]/\sigma_X))} \times$$

$$\left[1 - \Phi\left(\frac{l - \mu_X - \rho\sigma_X(y - \mu_Y)/\sigma_Y}{\sigma_X\sqrt{1 - \rho^2}}\right)\right],$$

(4)

where $f_{Y,X}(\cdot)$ is the joint density function of $X$ and $Y$. 

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The CDF of $[Y|X > l]$ is

$$F_{[Y|X > l]}(y; \theta) = \int_{-\infty}^{y} f_{[Y|X > l]}(y; \theta) dy$$

$$= \int_{-\infty}^{y} \frac{\exp \left\{ -(y - \mu_Y)^2 / (2\sigma_Y^2) \right\}}{\sigma_Y \sqrt{2\pi}} \times \left[ 1 - \Phi \left( \frac{l - \mu_X - \rho \sigma_X (y - \mu_Y) / \sigma_Y}{\sigma_X \sqrt{1 - \rho^2}} \right) \right] dy.$$  \hspace{1cm} (5)

The theoretical quantile function of $[Y|X > l]$ is the inverse function of $F_{[Y|X > l]}(y; \theta)$, denoted by $Q_{[Y|X > l]}(p; \theta)$ where $p$ is a constant and $0 < p < 1$. We solve for the $y$ such that $F_{[Y|X > l]}(y; \theta) = p$ by solving $F_{[Y|X > l]}(y; \theta) - p = 0$ using the Newton-Raphson method. Then, the quantile function evaluated at that $p$ is $Q_{[Y|X > l]}(p; \theta) = y$.

Suppose we conduct a SPLD experiment that loads the specimens in the $X$–mode with a proof load level, and then tests the survivors to failure in the $Y$–mode. The proof load level is low enough to ensure survivors suffer little or no damage. For example, the 5th percentile of the strength distribution is generally considered to be low enough to insure insignificant damage in the strength of the proof load survivors. The SPLD–with–shoulder approach would allow us to obtain the maximum likelihood estimate (MLE) of $\theta$, denoted by $\hat{\theta}^{MLE}$. Then for any proof load level $l$, the approximated quantile function is $Q_{[Y|X > l]}(p; \hat{\theta}^{MLE})$.

Suppose that a second SPLD experiment is conducted. The experiment proof loads the specimens in the $X$–mode up to a proof load level $l_2$. The survivors are tested to failure in the $Y$–mode. We are concerned about the damage accumulated in the strength of the proof load survivors. The survivor observations are denoted as $\{y^*_i : i = 1, 2, \ldots, n_y^*\}$. The empirical distribution $p_i$ of each $y^*_i$ with a continuity correction is

$$p_i = \frac{\text{sum}(I\{y^*_i \leq y^*_i : i = 1, 2, \ldots, n_y^*\}) - 0.5}{n_y^*}.$$  

The estimated theoretical quantiles of $[Y|X > l_2]$ is $Q_{[Y|X > l_2]}(p_i; \hat{\theta}^{MLE})$ for $i = 1, 2, \ldots, n_y^*$.

We plot the empirical quantile $y^*_i$ against the estimated theoretical quantile $Q_{[Y|X > l_2]}(p_i; \hat{\theta}^{MLE})$ for $i = 1, 2, \ldots, n_y^*$ (Q–Q plot). If the proof loading procedure does not damage the survivors, the empirical quantile of $[Y^*|X > l_2]$ should be similar to the estimated theoretical quantile of
The Q–Q line is expected to stay close to the 45–degree line. Any departure from the 45–degree line is due to the damage caused by proof loading as simulation studies of the SPLD–with–shoulder approach indicated that $\hat{\theta}^{MLE}$ is accurate (Cai 2015a).

Simulation studies are conducted to verify that the proposed Q–Q line reveals the damage as expected on the reasoning above, and the results are reported in the next section.

Simulation studies In simulation studies, we choose a damage model $[Y^*|X > l_2] = [Y|X > l_2] - \alpha/[Y|X > l_2]$ where $\alpha \geq 0$. This damage model reflects our intuition and preliminary data analysis of the damage accumulated in the strength of the proof load survivors based on our Summer–of–2011 experiment data (details later): more severe for weaker specimens and less severe for strong specimens. Simulation studies in Parts I and II treat the parameter $\theta$ as known, but Part III treats $\theta$ as unknown.

Part I A SPLD sample of size $N$ is simulated from a bivariate normal distribution with $\mu_X = 6.48, \sigma_X = 1.85, \mu_Y = 1.40, \sigma_Y = 0.4$, and $\rho = 0.2$. The specimens are proof loaded in the $X$–mode up to a load level $l_2$ specified so that 50% of the specimens fail below the load level. We observe $X_i$ if $X_i$ is smaller than the proof load level $l_2$, and observe $Y_i^* = Y_i - \alpha/Y_i$ if $X_i > l_2$. The observations on the survivors are $\{y_1^*, y_2^*, \ldots, y_{n_y^*}\}$ where $n_y^* = N/2$. We calculate the empirical CDF of $Y^*$ for each $y_i^*$, denoted as $p_i$. The theoretical quantile $Q_{[Y|X > l_2]}(p_i; \theta)$ is calculated for each $p_i$ where we take $\theta$ as the true value. The reason why we take $\theta$ as the true value for calculating the quantile $Q_{[Y|X > l_2]}(p_i; \theta)$ is because the goal of the simulation studies in part I is to verify whether the proposed Q–Q line in Section 2.3 resembles the damage function. Note that in practice, good estimates of $\theta$ can be obtained by the SPLD–with–shoulder approach ((Amorim and Johnson 1986; Cai 2015b)). The proposed Q–Q line for revealing the damage information is produced by plotting $y_i^*$ against $Q_{[Y|X > l_2]}(p_i; \theta)$ for $i = 1, 2, \ldots, n_y^*$. Figure 1 presents the Q–Q plots for the cases where $\alpha = 0$ with $N = 100, 250, 500, 1000$. When $\alpha = 0$, no damage occurs on the survivors and so the empirical quantiles are expected to resemble the theoretical quantiles. In other words, the Q–Q line should lie close to the 45–degree line. In fact, each Q–Q line in Figure 1 lies close to the 45–degree line (the solid line), in agreement with theory and confirms that survivors have not been damaged. Similarly, Figure 2 presents the Q–Q
plots for the cases when $\alpha = 0.1$ with $N = 100, 250, 500, 1000$. Figure 3 presents the Q–Q plots for the cases when $\alpha = 0.5$ with $N = 100, 250, 500, 1000$. The solid lines in Figures 2 and 3 represent the true damage model. The corresponding Q–Q line resembles the true damage model quite well.

Figure 1: Q–Q plot: $y_i^*$ against $Q_{Y|X>l_2}(p_i; \theta)$ for $\alpha = 0$ and $N = 100, 250, 500, 1000$. The solid line is the real damage model $Y^* = Y$. The points lie on the 45 degrees line as they should. There is no damage.

Part II A SPLD sample of size $N = 300$ is simulated from a bivariate normal distribution with $\mu_X = 6.48, \sigma_X = 1.85, \mu_Y = 1.40, \sigma_Y = 0.4$, and $\rho$ is either 0.2 or 0.8. The specimens are proof loaded in the $X$–mode up to a load level $l_2$. The load level is specified so that 50% of the specimens fail below the load level. We observe $X_i$ if $X_i$ is smaller than the proof load level $l_2$, and observe $Y_i^* = Y_i - \alpha/Y_i$ if $X_i > l_2$. The specification of $\alpha$ is guided by our Summer–of–2011 experiment data. Realistic values of $\alpha$ should be specified such that $P(Y^* > 0.51|X > l_2) = P(Y - \alpha/Y > 0.51|X > l_2)$ has a high probability (say 0.90) when $\rho$ is large and positive. Therefore, the simulation studies
Figure 2: Q–Q plot: $y_i^*$ against $Q_{\{Y|X>l_2\}}(p_i; \theta)$ for $\alpha = 0.1$ and $N = 100, 250, 500, \text{ or } 1000$. The solid line is the damage model $Y^* = Y - 0.1/Y$. The points lie closely to the true damage model.
Figure 3: Q–Q plot: $y_i^*$ against $Q_{[Y|X>u_2]}(p_i; \theta)$ for $\alpha = 0.5$ and $N = 100, 250, 500, \text{ or } 1000$. The solid line is the real damage model $Y^* = Y - 0.5/Y$. More severe damage occurs. The plots correctly reflect the change in the $Y$ values.
specify $\alpha$ to be 0, 0.1, 0.3, 0.5 or 0.7 when $\rho = 0.8$. When $\rho = 0.2$, the correlation between $X$ and $Y$ is weak. Proof loading a specimen in the $X$–mode should only cause slight or negligible damage in its strength in the $Y$–mode. Thus, $\alpha$ is specified to be 0, 0.05, 0.10, 0.15 or 0.20 when $\rho = 0.2$.

No damage occurs on the survivors when $\alpha = 0$.

The simulated observations on the survivors are denoted as

$$\{y_1^*, y_2^*, \ldots, y_{n_y^*}\}$$

where $n_y^* = N/2 = 150$. We calculate the empirical CDF of $Y^*$ for each $y_i^*$, denoted as $p_i$. The theoretical quantile $Q_{[Y|X>l]}(p_i; \theta)$ is calculated for each $p_i$ where $\theta$ is taken as the true value. The least squares estimate of $\alpha$ is found by minimizing the sum of squares of the errors,

$$\sum_{i=1}^{n_y^*}(y_i^* - Q_{[Y|X>l]}(p_i; \theta)) + \alpha/Q_{[Y|X>l]}(p_i; \theta))^2.$$ 

Thus, the least squares estimate of $\alpha$ is

$$\hat{\alpha}_{LS} = -\frac{\sum_{i=1}^{n_y^*}(y_i^* - Q_{[Y|X>l]}(p_i; \theta))}{\sum_{i=1}^{n_y^*}(1/Q_{[Y|X>l]}(p_i; \theta))^2}.$$ 

The above procedure is repeated $10^4$ times. Table 1 reports the average and the standard deviation of the $10^4$ least squares estimates of $\alpha$. For each simulation case, the average of the least squares estimates stays in a close neighbourhood of the true value. The simulation results suggest that the least squares estimate of $\alpha$ is accurate.

<table>
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<th>EST</th>
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<th>EST</th>
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</table>

Part III The simulation studies in Parts I treat the parameter $\theta$ as known. In Part II, the parameter $\theta$ is treated as unknown. One additional SPLD sample is generated to provide an estimate of $\theta$ using the SPLD–with–shoulder approach.

A sample of size 1000 for the SPLD–with–shoulder approach is simulated based on a bivariate normal distribution. The parameters of the bivariate normal distribution are $\mu_X = 6.48, \sigma_X = 1.85, \mu_Y = 1.43, \sigma_Y = 0.40$ and $\rho$ is either 0.2 or 0.8. Following the optimal allocation suggested by
(Cai 2015b), we assign 50% of the specimens to the SPLD group and the rest to the shoulder group. The shoulder group of size 500 (1000 × 0.5) is simulated from the univariate normal distribution with \( \mu_Y = 1.43 \) and \( \sigma_Y = 0.40 \). For the SPLD group of size 500, specimens are proof loaded in the \( X \)-mode. The proof load level is specified such that 20% of the specimens fail below the load level. In practice, we will also use a low proof load level for safely assuming that the damage accumulated in the proof load survivors is negligible. The proof load survivors are tested to failure in the \( Y \)-mode.

We calculate the MLE of \( \theta \) that optimizes the likelihood given the generated SPLD–with–shoulder sample, and denote it as \( \hat{\theta}^{MLE} \).

Next, we generate a second SPLD sample based on the same bivariate normal distribution. The sample size is 300. The specimens are proof loaded in the \( X \)-mode up to a load level \( l_2 \) specified so that 50% of the specimens fail below the load level. We observe \( X_i \) if \( X_i \) is smaller than \( l_2 \), and observe \( Y_i^* = Y_i - \alpha/Y_i \) otherwise. The simulation studies specify \( \alpha \) to be 0, 0.1, 0.3, 0.5 or 0.7 when \( \rho = 0.8 \). When \( \rho = 0.2 \), \( \alpha \) is specified to be 0, 0.05, 0.10, 0.15, and 0.20. No damage occurs on the survivors when \( \alpha = 0 \). The generated observations on the survivors are denoted as \( \{y_1^*, y_2^*, \ldots, y_{n_y^*}\} \) where \( n_{y^*} = 150 (300/2) \).

We calculate the empirical CDF of \( [Y^*|X > l_2] \) for each \( y_i^* \), denoted as \( p_i \). The estimated theoretical quantile \( Q_{[Y|X>l_2]}(p_i; \hat{\theta}^{MLE}) \) is calculated for each \( p_i \). Given the generated observations \( y_i^* \)'s and the estimated theoretical quantiles \( Q_{[Y|X>l_2]}(p_i; \hat{\theta}^{MLE}) \)'s, we estimate \( \alpha \) by the least squares method that minimizes the sum of the squares of the errors,

\[
\sum_{i=1}^{n_{y^*}} (y_i^* - Q_{[Y|X>l_2]}(p_i; \hat{\theta}^{MLE})) + \alpha / Q_{[Y|X>l_2]}(p_i; \hat{\theta}^{MLE})^2,
\]

with respect to \( \alpha \). Thus, the approximate least squares estimate of \( \alpha \) is

\[
\hat{\alpha}_{LS} = -\frac{\sum_{i=1}^{n_{y^*}} (y_i^* - Q_{[Y|X>l_2]}(p_i; \hat{\theta}^{MLE}))/Q_{[Y|X>l_2]}(p_i; \hat{\theta}^{MLE})}{\sum_{i=1}^{n_{y^*}} (1/Q_{[Y|X>l_2]}(p_i; \hat{\theta}^{MLE}))^2},
\]

where the double hat notation means that this is an estimate of an estimate derived by plugging in \( \hat{\theta}^{MLE} \) for the unknown parameter \( \theta \).

The above procedure is repeated \( 10^4 \) times. Table 2 reports the average and the standard
deviation of the $10^4$ approximate least squares estimate of $\alpha$. The average of the approximate least squares estimates stays in a close neighbourhood of the true value, which indicates that our empirical approach provides an accurate estimate of $\alpha$. When $\rho = 0.2$, our study suggests its estimator is unbiased but the coefficient of variation in the $\alpha$ estimate is rather large. The sample sizes required to detect significant damage when $\rho$ is small should be investigated. However, assessing the proof load damage when $X$ and $Y$ are weakly correlated will not of much practical interest because the method would only be used in situations where it would known that the strength properties are related. Table 2 also reports the average and the standard deviation of the $10^4$ MLE’s of $\theta$ to demonstrate that the accuracy of the MLE of $\theta$ performs reasonably well.

Table 2: Unknown $\theta$ : performance of the least squares estimate of $\alpha$. The simulation setting is described as in the text.

<table>
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Summary Our simulation studies in Part I demonstrate that comparing the estimated theoretical quantiles of the undamaged strength $[Y | X > l_2]$ with the empirical quantiles of the survivor strength...
\[ [Y^* | X > l_2] \text{ (Q–Q plot) resembles the true damage model well. In addition, simulation studies in Parts II and III suggest that the damage model parameter } \alpha \text{ can be estimated accurately by the least squares estimation method. We suggest only using our method when } X \text{ and } Y \text{ are known to be related because the coefficient of variation of } T \text{ the } \alpha \text{ estimate could be rather large when the correlation between } X \text{ and } Y \text{ is weak.} \]

3. BREAKING THE SAME BOARD TWICE

This section will show how the damage modelling approach presented in Subsection 2.3 can be used to find that maximum likelihood estimator of the correlation between two destructively measured strength properties \( X \) and \( Y \) that have a bivariate normal distribution. That is

\[
\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2 \left( \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma^2_X & \rho \sigma_X \sigma_Y \\ \rho \sigma_X \sigma_Y & \sigma^2_Y \end{bmatrix} \right).
\]

The data are assumed to be generated by an SPLD–with–shoulder design where specimens are proof loaded in the \( X \– \text{mode up to a pre–determined proof load level } l \). The survivors are then tested to failure in the \( Y \– \text{mode. If the survivors were not damaged, we would observe two random variables } Z \text{ and } U \text{ given by}

\[
Z = \begin{cases} X & \text{if } X < l \\ Y & \text{if } X > l \end{cases} \quad \text{and } \quad U = \begin{cases} 1 & \text{if } X < l \\ 0 & \text{if } X > l \end{cases}.
\]

The factor in the likelihood function for \( \theta = (\mu_X, \sigma_X, \mu_Y, \sigma_Y, \rho)^T \), given by the observation \( s = (z, u) \) from the SPLD experiment alone would be

\[
L(\theta; s) = \{f_X(x)\}^u \{f_{Y|X>l}(y)P(X > l)\}^{1-u} = \left\{ \frac{\exp \left[ -\frac{(x-\mu_X)^2}{2\sigma_X^2} \right]}{\sigma_X} \right\}^u \left\{ \frac{\exp \left[ -\frac{(y-\mu_Y)^2}{2\sigma_Y^2} \right]}{\sigma_Y} \right\} \int_{a_l}^{\infty} e^{-a^2/2} da \right\}^{1-u},
\]

where \( f_X(\cdot) \) is the density of \( X \), \( f_{Y|X>l}(\cdot) \) is the conditional density of \( Y \) given \( X > l \), and

\[ a_l = \{l - \mu_X - \rho(\sigma_X/\sigma_Y)(y - \mu_Y)\} / \{\sigma_X \sqrt{1 - \rho^2}\}. \]

The factor in the joint likelihood function of \( \theta \)
given by the observation \( s = (z, u) \) from the SPLD experiment augmented by an observation \( y \) from the shoulder group would be

\[
L_{PY}^{*}(\theta; s, y) = L(\theta; s)L_Y(\mu_y, \sigma_y; y),
\]

(7)

where \( L_Y(\mu_y, \sigma_y; y) = \exp\left(-\frac{(y - \mu_y)^2}{2\sigma_y^2}\right)/(\sigma_y\sqrt{2\pi}) \). The full likelihood would then be the appropriate product of these factors and from it, the MLE \( \hat{\theta} \) of \( \theta \) could be obtained.

However were the proof load level set sufficiently large as to cause the specimen to be damaged say to \( l_2 \), we would observe \( (Z^*, U) \) instead of \( (Z, U) \) in the survivors, given by

\[
Z^* = \begin{cases} 
X & \text{if } X < l_2 \\
Y^* & \text{if } X > l_2 
\end{cases} \quad \text{and} \quad U = \begin{cases} 
1 & \text{if } X < l_2 \\
0 & \text{if } X > l_2 ,
\end{cases}
\]

where \( Y^* \) would be the residual strength in the \( Y \)-mode Given the observation \( s^* = (z^*, u) \) from the SPLD experiment and the observation \( y \) from the shoulder group of \( Y \), the factor in the joint likelihood function for \( \theta \) would be

\[
L_{PY}^{*}(\theta; s^*, y) = L^*(\theta; s^*)L_Y(\mu_y, \sigma_y; y),
\]

where \( L^*(\theta; s^*) \) is the likelihood function of \( \theta \) given the observation \( s^* = (z^*, u) \). Thus,

\[
L^*(\theta; s^*) = \{f_X(x)\}^u\{f_{Y^*|X>l_2}(y^*)P(X > l_2)\}^{1-u},
\]

where \( f_{Y^*|X>l_2}(\cdot) \) is the conditional density function of \( Y^* \) given \( X > l_2 \).

The density function of \( [Y^*|X > l_2] \) is unknown. So, the likelihood function \( L_{PY}^{*}(\theta; s^*, y) \) is unknown when we observe \( s^* = (z^* = y^*, u = 0) \). If the deterministic relationship between the random variables \( [Y^*|X > l_2] \) and \( [Y|X > l_2] \) were available, we could derive the density function of \( [Y^*|X > l_2] \) by implementing change of variables on the density function of \( [Y|X > l_2] \) using their deterministic relationship. However, only the distributional relationship between \( [Y^*|X > l_2] \) and \( [Y|X > l_2] \) is assessable. The relationship between \( [Y^*|X > l_2] \) and \( [Y|X > l_2] \) is found by comparing the empirical quantiles of \( [Y^*|X > l_2] \) with the estimated theoretical quantiles of
[\{Y|X > l_2\}] as seen in Section 2.2 e.g. by their Q–Q plot. But the structural relationship between
\([Y^*|X > l_2]\) and \([Y|X > l_2]\) cannot be ascertained from that plot, since it represents just the
distributional relationship between these two distributions.

In short, when the proof load level is set high enough to induce damage in the proof load
survivors, we observe only \(s^*\) for the unknown likelihood function \(L_{PFY}(\theta; s^*, y)\). But we do not
have the observation \(s\) we need with likelihood function \(L_{PFY}(\theta; s, y)\).

This section addresses this problem by predicting the observation \(s\) from \(s^*\). More precisely, a
sample from \([Y|X > l_2]\) is generated from the sample from \([Y^*|X > l_2]\) using the estimated relation-
ship between their distributions. Once the needed observations \(s\) for the likelihood
\(L_{PFY}(\theta; s, y)\) are in hand, we can get an MLE for \(\theta\) by maximizing the joint likelihood function
\(L_{PFY}(\theta; s, y)\) with respect to \(\theta\).

3.1 Predicting \(s\) from \(s^*\)

This section shows how a sample from the distribution of \([Y|X > l_2]\) is generated based on a sample
from the distribution of \([Y^*|X > l_2]\) using their distributional relationship. The method assumes the
equality in distribution of random variables \([h(Y)|X > l_2]\) and \([Y^*|X > l_2]\), where \(h\) is a continuous
and invertible function.

In our lumber application in Section 4 \(h\) is specified as the empirically assessed proof load damage
model \([h(Y)|X > l_2] = [Y|X > l_2] - \alpha/[Y|X > l_2]\) where \(\alpha \geq 0\). If \([h(Y)|X > l_2]\) and \([Y^*|X > l_2]\)
are equal in distribution, then

\[F_{[h(Y)|X > l_2]}(h(y)) = F_{[Y^*|X > l_2]}(y^*),\]

where \(F_{[h(Y)|X > l_2]}(\cdot)\) and \(F_{[Y^*|X > l_2]}(\cdot)\) are the CDFs of \([h(Y)|X > l_2]\) and \([Y^*|X > l_2]\), respectively.

Both \([h(Y)|X > l_2]\) and \([Y^*|X > l_2]\) are continuous random variables, and thus their CDFs are
invertible.

Let \(s^* = \{x_1, x_2, \cdots, x_{k_s}, y_1^*, y_2^*, \cdots, y_{n-k_s}^*\}\) be the observed sample of size \(n\) from a SPLD
experiment where specimens are proof loaded in the \(X\)-mode up to the load level \(l_2\), and the
survivors are tested to failure in the \(Y\)-mode. The observations \(\{x_1, x_2, \cdots, x_{k_s}\}\) are collected
from the \(k_s\) specimens that fail below the proof load level \(l_2\), while \(\{y_1^*, y_2^*, \cdots, y_{n-k_s}^*\}\) are the
observations collected from the $n - k_s$ damaged proof load survivors. Since \( \{y_1^s, y_2^s, \ldots, y_{n-k_s}^s\} \) is an observed sample from \( Y^* | X > l_2 \), the sample \( \{u_i = F_{Y^* | X > l_2}(y_i^s) : i = 1, 2, \ldots, n - k_s\} \) is an observed sample from the uniform distribution with parameters 0 and 1. We generate observations \( \{y_i : i = 1, 2, \ldots, n - k_s\} \) by specifying \( y_i = h^{-1}(F_{Y^* | X > l_2}^{-1}(u)) = h^{-1}(y_i^s) \) where \( h^{-1}(\cdot) \) is the inverse function of \( h(\cdot) \). Since

\[
\Pr(h^{-1}(F_{Y^* | X > l_2}^{-1}(u)) \leq y) = \Pr(F_{Y^* | X > l_2}^{-1}(u) \leq h(y)) \\
= \Pr(u \leq F_{Y^* | X > l_2}(h(y))) \\
= F_{Y^* | X > l_2}(h(y)) \\
= \Pr(Y^* \leq h(y)|X > l_2) \\
= \Pr(d(Y) \leq h(y)|X > l_2) \\
= \Pr(Y \leq y|X > l_2) \\
= F_{Y | X > l_2}(y),
\]

the generated sample \( \{y_i : i = 1, 2, \ldots, n - k_s\} \) is an sample from the distribution of \( Y|X > l_2 \).

Based on this, we specify the desired sample \( s \) for the likelihood \( L_{PFY}(\theta; s, y) \) to be

\( \{x_1, x_2, \ldots, x_{k_s}, y_1, y_2, \ldots, y_{n-k_s}\} \) where \( y_i = h^{-1}(y_i^s) \). The parameters of interest are estimated by the MLE that maximizes the joint likelihood function \( L_{PFY}(\theta; s, y) \) with respect to \( \theta \) given the generated sample \( s = \{x_1, x_2, \ldots, x_{k_s}, y_1, y_2, \ldots, y_{n-k_s}\} \) and a sample \( y \) from the shoulder group of \( Y \). Simulation were conducted to demonstrate the success of the approach, since an analytical approach in this complex situation is not feasible (Cai 2015a). These are presented in the next section.

3.2 Simulation studies

The simulation studies in this section investigate the MLE performance obtained by the generalized SPLD–with–shoulder approach. The simulation studies contain two parts. In both parts, we consider the distributional relationship between \( Y^* | X > l_2 \) and \( Y|X > l_2 \) to be the empirical damage model for the proof load effect presented in Chapter ???. The damage model is

\( [Y^*|X > l_2] \overset{d}{=} [Y|X > l_2] - \alpha /[Y|X > l_2] \) where \( \overset{d}{=} \) means equal in distribution. The first part of
the simulation studies (Part I) treats the parameter $\alpha$ as known. The second part of the simulation studies (Part II) treats the parameter $\alpha$ as unknown, and estimates $\alpha$ using the empirical approach of quantifying the proof load damage presented in Section 2.3, Chapter ???. For estimating $\alpha$, this empirical approach needs the theoretical quantiles of $Y$ conditional on $X > l_2$. Those theoretical quantiles are estimated in Part II by generating an additional SPLD sample.

For each part of the simulation studies, the accuracy of the MLE obtained by the generalized SPLD–with–shoulder approach is inspected. The precision performance of the MLE is also inspected as additional uncertainty is introduced.

Part I: The simulation studies in Part I treat the parameter $\alpha$ as known.

Part I: data simulation A SPLD–with–shoulder sample is generated from a bivariate normal distribution. The parameters of the bivariate normal distribution are $\mu_x = 6.48$, $\sigma_x = 1.85$, $\mu_y = 1.43$, $\sigma_y = 0.40$ and $\rho$ is either 0.5 or 0.8. The moderate to high correlations are considered because assessing the proof load damage and then applying the generalized SPLD–with–shoulder approach is not of practical interest when $X$ and $Y$ are weakly correlated. The total sample size is $N = 600$. The specimen allocation between the SPLD group and the shoulder group is chosen based on the simulation studies of the optimal allocation in Section 3.3, Chapter 3. We assign 50% of the specimens to the SPLD group and the rest to the shoulder group. The proof load level of the SPLD group is chosen based on the simulation studies of the optimal proof load level in Section 3.3, Chapter 3. So the proof load level $l_2$ of the SPLD group is specified such that 80% of the specimens fail below the load level. We generate the SPLD sample such that $[Y|X > l_2] - \alpha/[Y|X > l_2]$ and $[Y^*|X > l_2]$ are equal in distribution but are not equal deterministically. The SPLD sample size is 300 ($N \times 0.5 = 300$). The sample for the shoulder group is generated from the univariate normal distribution with $\mu_y = 1.43$ and $\sigma_y = 0.40$. The shoulder sample of size 300 ($N \times 0.5 = 300$) is denoted by $\{y_i, i = 1, 2, \cdots, 300\}$.

To simulate the SPLD sample $s^*$ such that $[Y|X > l_2] - \alpha/[Y|X > l_2]$ and $[Y^*|X > l_2]$ are equal in distribution but are not equal deterministically, we simulate two SPLD samples of size 300 each and only take the desired observations from each SPLD sample (details below).

The first SPLD sample of size 300 is generated from the same bivariate normal distribution
where the parameters are $\mu_x = 6.48, \sigma_x = 1.85, \mu_y = 1.43, \sigma_y = 0.40$ and $\rho$ is either 0.5 or 0.8. The proof load level $l_2$ is specified in the $X$–mode such that 80% of the specimens fail below the load level. The generated sample of size 300 is \{(x_{1i}, y_{1i}) : i = 1, 2, \cdots, 300\}. All $x$ observations are kept if the $x$ observation is small than the proof load level $l_2$. Equivalently, the $x$ observations are kept where $\{x_{1i} : x_{1i} < l_2, i = 1, 2, \cdots, 300\}$. The size of $\{x_{1i} : x_{1i} < l_2, i = 1, 2, \cdots, 300\}$ is 240 ($N \times 0.5 \times 0.8 = 240$). All other generated observations including the $y$ observations are discarded.

The second SPLD sample is generated from the same bivariate normal distribution. The bivariate normal sample is \{(x_{2i}, y_{2i}) : i = 1, 2, \cdots, 300\}. However, this time, we take all $y$ observations where the corresponding $x$ observation is greater than the proof load level $l_2$. Equivalently, the $y$ observations are kept where $\{y_{2j} : x_{2j} > l_2, j = 1, 2, \cdots, 300\}$. The size of $\{y_{2j} : x_{2j} > l_2, j = 1, 2, \cdots, 300\}$ is 60 ($N \times 0.5 \times 0.2 = 60$). Each selected $y$ is transformed to a residual strength $y^\star$ by $y^\star = y - \alpha/y$. The specification of $\alpha$ is guided by the data values from the Summer–of–2011 experiment. The minimum of the observed log–transformed $UTS$ value is 0.51 from the R40 proof loaded group, which corresponds to the 1st empirical quantile of the log-transformed $UTS$ distribution. Realistic values of $\alpha$ should be specified such that $P(Y^\star > 0.51|X > l_2) = P(Y - \alpha/Y > 0.51|X > l_2)$ has a high probability (say 0.90) when the positive correlation between $X$ and $Y$ is strong. Therefore, the simulation studies specify $\alpha$ to be 0, 0.1, 0.3, 0.5 or 0.7. No damage occurs on the survivors when $\alpha = 0$.

The simulated $s^\star$ sample is the union of the two samples $\{x_{1i} : x_{1i} < l_2, i = 1, 2, \cdots, 300\} \cup \{y_{2j}^\star : x_{2j} > l_2, j = 1, 2, \cdots, 300\}$. The sample $\{y_{2j}^\star : x_{2j} > l_2, j = 1, 2, \cdots, 300\}$ satisfies the desired simulation setting where $[Y^\star|X > l_2]$ and $[Y|X > l_2] - \alpha/[Y|X > l_2]$ are equal in distribution, but are not equal deterministically.

Up to this point, we have simulated a shoulder sample $y$ and a SPLD sample $s^\star$ where the proof load survivors are damaged. The generated sample for investigating the generalized SPLD–with–shoulder approach is $\{s^\star \cup y\}$.

Part I: implementation of the simulated data The desired sample $s$ for the likelihood $LFY(\theta; \cdot)$ is specified to be $\{x_{1i} : x_{1i} < l_2, i = 1, 2, \cdots, 300\} \cup \{y_{2j} : x_{2j} > l_2, j = 1, 2, \cdots, 300\}$ where $y_{2j} = h^{-1}(y_{2j}^\star)$ and $h(y_{2j}) = y_{2j} - \alpha/y_{2j}$. We calculate the MLE of $\theta$ that maximizes the joint
likelihood \( L_{PFY}(\theta; s, y) \) given the generated observations \( y = \{y_i, i = 1, 2, \cdots, 300\} \) and \( s = \{x_i : x_1 < l_2, i = 1, 2, \cdots, 300\} \cup \{y_2 : x_2 > l_2, j = 1, 2, \cdots, 300\} \).

The above procedures are repeated 10,000 times. The average and the standard deviation of the MLE’s across 10,000 data sets are summarized in Table 3. The average of the MLE’s stays in a close neighbourhood of the true value, indicating that the MLE of \( \theta \) is an accurate estimate.

Table 3: Known \( \alpha \) with a sample size \( N = 600 \): performance of the MLE obtained by the generalized SPLD–with–shoulder approach. The damage model is \( [Y^* | X > l_2] \overset{d}{=} [Y | X > l_2] - \alpha / [Y | X > l_2] \). Simulation details are described in the text.

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We conduct the above simulation studies (data simulation and implementation of simulated data) again, but with a smaller sample size \( N = 300 \). The results are summarizes in a similar fashion as Table 3, and are reported in Table 4. The MLE of \( \theta \) remains accurate even for the smaller sample size \( N = 300 \).
Table 4: Known $\alpha$ with a sample size $N = 300$: performance of the MLE obtained by the generalized SPLD–with–shoulder approach. The damage model is $[Y^*|X > l_2] = [Y|X > l_2] - \alpha/[Y|X > l_2]$. Simulation details are described in the text.

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Part II: The simulation studies in Part II treat the parameter $\alpha$ as unknown, and estimate it by the empirical approach of quantifying proof load damage presented in Section 2.3, Chapter ??.

Part II: data simulation A SPLD–with–shoulder sample of size $N_1 = 1200$ is simulated based on a bivariate normal distribution. The parameters of the bivariate normal distribution are $\mu_x = 6.48, \sigma_x = 1.85, \mu_y = 1.43, \sigma_y = 0.40$ and $\rho$ is either 0.5 or 0.8. Following the optimal allocation developed in Chapter 3, we assign 50% of the specimens to the SPLD group and the rest to the shoulder group. The shoulder group of size 600 ($N_1 \times 0.5$) is simulated from the univariate normal distribution with $\mu_y = 1.43$ and $\sigma_y = 0.40$. For the SPLD group of size 600 ($N_1 \times 0.5$), specimens are proof loaded in the $X$–mode. The proof load level is specified such that 20% of the specimens fail below the load level. In practice, we will also use a low proof load level for safely assuming that the damage accumulated in the proof load survivors is negligible. The survivors are tested to failure in the $Y$–mode. We calculate the MLE of $\theta$ that optimizes the likelihood function $L_{PFY}(\theta; \cdot)$ (Equation 7) given the generated SPLD–with–shoulder sample, and denote it as $\hat{\theta}^{MLE}$.

Next, we generate a second SPLD–with–shoulder sample where the proof load survivors are damaged. The sample size is $N_2 = 600$. We assign 50% of the specimens to the SPLD group and the rest to the shoulder group of $Y$. The proof load level $l_2$ of the SPLD group is specified such that we expect 80% of the specimens are failed below the proof load level. The SPLD sample $s^*$ is generated such that $[Y|X > l_2] - \alpha/[Y|X > l_2]$ and $[Y^*|X > l_2]$ are equal in distribution but are not equal deterministically. The shoulder sample of size 300 ($N_2 \times 0.5$) is generated from the univariate normal distribution with $\mu_y = 1.43$ and $\sigma_y = 0.40$. The shoulder sample is denoted by $y = \{y_i, i = 1, 2, \cdots, 300\}$.

To simulate the sample $s^*$ of size 300 such that $[Y|X > l_2] - \alpha/[Y|X > l_2]$ and $[Y^*|X > l_2]$ are equal in distribution but are not equal deterministically, we simulate two SPLD samples of size 300 each and only take the desired observations from each proof loaded sample (details below).

The first SPLD sample of size 300 is generated from the same bivariate normal distribution where the parameters are $\mu_x = 6.48, \sigma_x = 1.85, \mu_y = 1.43, \sigma_y = 0.4$ and $\rho$ is either 0.5 or 0.8. The proof load level $l_2$ is specified in the $X$–mode such that 80% of specimens fail below the load level. The generated sample of size 300 is $\{(x_i, y_i) : i = 1, 2, \cdots, 300\}$. All $x$ observations are
kept if the \( x \) observation is small than the proof load level \( l_2 \). Equivalently, the \( x \) observations are kept where \( \{x_{1_i} : x_{1_i} < l_2, i = 1, 2, \cdots, 300\} \). The size of \( \{x_{1_i} : x_{1_i} < l_2, i = 1, 2, \cdots, 300\} \) is 240 \((N_2 \times 0.5 \times 0.8 = 240)\). All other generated observations including the \( y \) observations are discarded.

The second SPLD sample is generated from the same bivariate normal sample. The bivariate normal sample is \((x_{2_i}, y_{2_i}) : i = 1, 2, \cdots, 300\). However, this time, we take all \( y \) observations where the corresponding \( x \) observation is greater than the proof load level \( l_2 \). Equivalently, the \( y \) observations are kept where \( \{y_{2_j} : x_{2_j} > l_2, j = 1, 2, \cdots, 300\} \). The size of \( \{y_{2_j} : x_{2_j} > l_2, j = 1, 2, \cdots, 300\} \) is 60 \((N_2 \times 0.5 \times 0.2 = 60)\). Each selected \( y \) is transformed to a residual strength \( y^* \) by \( y^* = y - \alpha/y \). The simulation studies specify \( \alpha \) to be 0, 0.1, 0.3, 0.5 or 0.7. No damage occurs on the survivors when \( \alpha = 0 \).

The simulated \( s^* \) sample is the union of the two samples \( \{x_{1_i} : x_{1_i} < l_2, i = 1, 2, \cdots, 300\} \cup \{y_{2_j} : x_{2_j} > l_2, j = 1, 2, \cdots, 300\} \). The simulated sample \( \{y_{2_j} : x_{2_j} > l_2, j = 1, 2, \cdots, 300\} \) provides the desired simulation setting where \( [Y^*|X > l_2] \) and \( [Y|X > l_2] - \alpha/[Y|X > l_2] \) are equal in distribution, but are not equal deterministically.

Up to this point, we have simulated a SPLD sample \( s^* \) where the proof load survivors are damaged and a shoulder sample \( y \). The sample for the generalized SPLD–with–shoulder approach is \( s^* \cup y \).

Part II: Implementation of the simulated data

The empirical quantile of \([Y^*|X > l_2]\) for each \( y_{2_j}^* \) is \( p_j \) where

\[
p_j = \frac{\text{sum}\{I\{y_{2_j}^* \leq y_{2_j}^* \} : j = 1, 2, \cdots, 60\} - 0.5}{60}.
\]

The estimated theoretical quantile of \([Y|X > l_2]\) is \( Q_{[Y|X > l_2]}(p_j; \hat{\theta}^{MLE}) \). The approximate least squares estimate of \( \alpha \) is calculated by

\[
\hat{\alpha}^{LS} = -\frac{\sum_{j=1}^{60} (y_{2_j}^* - Q_{[Y|X > l_2]}(p_j; \hat{\theta}^{MLE})) / Q_{[Y|X > l_2]}(p_j; \hat{\theta}^{MLE})}{\sum_{j=1}^{60} (1/Q_{[Y|X > l_2]}(p_j; \hat{\theta}^{MLE}))^2}.
\]

The desired sample \( s \) for the likelihood \( L_{PFY}(\theta; s, y) \) is specified to be \( \{x_{1_i} : x_{1_i} < l_2, i = 1, 2, \cdots, 300\} \cup \{y_{2_j} : x_{2_j} > l_2, j = 1, 2, \cdots, 300\} \) where \( y_{2_j} = \hat{h}^{-1}(y_{2_j}^*), \hat{h}(y) = y - \hat{\alpha}^{LS}/y, \) and \( \hat{h}^{-1}(\cdot) \) is the inverse function of \( \hat{h}(\cdot) \). To ensure that \( \hat{h}(\cdot) \) is invertible, \((y_{2_j}^*)^2 + 4\hat{\alpha}^{LS} \) requires to be
greater than or equal to 0 for all \( j = 1, 2, \cdots, 60 \). We specify \( N_1 \) large enough such that \( \hat{\theta}^{MLE} \) is not very far off from the true value. If \( \hat{\theta}^{MLE} \) is very far off from the true value, \( \hat{h}(\cdot) \) may not be invertible.

We calculate the MLE of \( \theta \) that maximizes the joint likelihood function \( L_{PFY}(\theta; s, y) \) given the observations \( y = \{ y_i, i = 1, 2, \cdots, 300 \} \) and \( s = \{ x_{1i} : x_{1i} < l_2, i = 1, 2, \cdots, 300 \} \cup \{ y_{2j} : x_{2j} > l_2, j = 1, 2, \cdots, 300 \} \).

The above procedure is repeated 10,000 times. Table 5 reports the average and the standard deviation of the MLE's of \( \theta \) across 10,000 data sets. When \( \rho = 0.5 \), the damage function \( \hat{h}(\cdot) \) is not invertible about six percent of the time when \( \alpha = 0 \). This percentage reduces to 3%, 0.5%, 0.06%, and 0 as \( \alpha \) increases to 0.1, 0.3, 0.5 and 0.7, respectively. When \( \rho = 0.8 \) and \( \alpha = 0 \), the damage function \( \hat{h}(\cdot) \) is not invertible for 3 cases. The non–invertible cases are removed from the 10,000 cases. The average of the MLE's stay in a close neighbourhood of the true value. The simulation results suggest that the MLE is an accurate estimate of \( \theta \). The mean and the standard deviation of the approximate least squares estimates of \( \alpha \) are also reported in Table 5. The average of the approximate least squares estimates of \( \alpha \) stays in a close neighbourhood of the true value, which indicates that the approximate least squares estimate of \( \alpha \) is accurate.

The above simulation studies (data simulation and implementation of simulated data) are conducted again, but with different sample sizes \( (N_1 = 1200, N_2 = 300) \). When \( \rho = 0.5 \), the damage function \( \hat{h}(\cdot) \) is not invertible about four percent of the time when \( \alpha = 0 \). This percentage reduces to 0.4% as \( \alpha \) increases to 0.7. When \( \rho = 0.8 \) and \( \alpha = 0 \) (\( \alpha = 0.1 \)), the damage function \( \hat{h}(\cdot) \) is not invertible for 4 cases (1 case). The non–invertible cases are removed from the 10,000 cases. The results are summarized in a similar fashion as Table 5, and are reported in Tables 6. The accuracy of the MLE of \( \theta \) remains reasonably well.

4. APPLICATION

In the summer of 2011, a large scale experiment was carried out by the Forest Products Stochastic Modelling Group located at the University of British Columbia. The experiment, hereafter referred to as the “Summer–of–2011 experiment” has been described in detail elsewhere (Cai, Cai, Chen, Golchi, Guan, Karim, Liu, Tomal, Xiong, Zhai, Lum, Welch and Zidek 2106).
Table 5: Unknown $\alpha$ with sample sizes $N_1 = 1200$ and $N_2 = 600$: performance of the MLE obtained by the generalized SPLD–with–shoulder approach. The damage model is $[Y^*|X > l_2] \overset{d}{=} [Y|X > l_2] - \alpha/[Y|X > l_2]$. Simulation details are described in the text.

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Table 6: Unknown $\alpha$ with sample sizes $N_1 = 1200$ and $N_2 = 300$: performance of the MLE obtained by the generalized SPLD–with–shoulder approach. The damage model is $[Y^*|X > l_2] \equiv [Y|X > l_2] - \alpha/[Y|X > l_2]$. Simulation details are described in the text.

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Briefly the experiment came in three bundles of lumber: two bundles labelled \#1 and \#2 were of grade – type SPF 1650f-1.5E, while one labelled \#3 was of type SPF No.2. Each 2 × 4 inch specimen was 12 feet long. SPF standards for Spruce, Pine and Fir species. The SPF 1650f-1.5E’s had a design value of 1650 psi for the fibre stress in bending and an expected average \( MOE \) of 1.5 million psi. The SPF No.2s were No.2 grade lumber that came from a mixture of Spruce, Pine, and Fir species. Both grades were produced together from the same mill. The information about the mill is confidential. We selected SPF because it is Canada’s highest volume species group. Compared with the two bundles of 1650f-1.5E, the SPF No.2 bundle came from a population with generally weaker lumber strength properties. The inclusion of the SPF No.2 bundle added variation to the lumber strength properties, making the combined sample resemble to some small extent, a sample from the global population of all lumber of that grade. However, in fact the lumber strengths came from a mixture distribution as we will see in the analysis described in the sequel.

The material was divided into eight balanced subsamples of varying size. The two shoulder subsamples T100 and R100 consisted of specimens that were tested to failure in \( UTS \)–mode and in the \( MOR \)–mode, respectively. In three subsamples, R20, R40, and R60, the specimens were first proof loaded in the \( MOR \)–mode to stress levels set so that respectively 20\%, 40\% and 60\% failed on being proof loaded. The survivors were then tested to failure in the \( UTS \)–mode. In a symmetrically opposite fashion, the remaining three proof loaded groups T20, T40, and T60 were proof loaded in the \( UTS \)–mode and the survivors tested to failure in the \( MOR \)–mode.

Combining each \( MOR \) proof–loaded subsample R20, R40 and R60 with the shoulder subsample T100 yielded a SPLD–with–shoulder design. Similarly subsamples T20, T40 and T60 when combined with R100 yielded a sample from a SPLD–with–shoulder design. Each shoulder subsample contained 174 specimens, while each of the six remaining groups had 87 specimens for a total of 870 specimens.

Both \( MOR \) and \( UTS \) are measured in pounds per square inch (psi). With that understanding, we non–dimensionalize them so that they are unitless quantities. With that understanding we let \( Y = \log UTS \) denote a unitless random variable that has approximately a Gaussian distribution, based on our exploratory data analyse. For similar reasons we let \( X = MOR \) and assume these two variables have an approximately bivariate normal distribution with means and standard deviations.
respectively \( \mu_x(\mu_y) \) and \( \sigma_x(\sigma_y) \). Denote the correlation between \( X \) and \( Y \) is \( \rho \) and the parameter vector be \( \mathbf{\theta} = (\mu_x, \sigma_x, \mu_y, \sigma_y, \rho)^T \). The following list summarizes the samples from the Summer–of–2011 experiment with proof load damage models obtained in Section 2.3:

- Subsample group 1–T20, R100 and T100. No damage on the T20 proof load survivors.
- Subsample group 2–T40 and R100. The proof load survivors are damaged. The damage model is \([X^*|Y > 1.38] \overset{d}{=} [X|Y > 1.38] - 1.55/[X|Y > 1.38]\). The desired likelihood sample \( s \) from the distribution of \([X|Y > 1.38]\) is generated based on the observed sample \( s^* \) from the distribution of \([X^*|Y > 1.38]\).
- Subsample group 3–T60 and R100. No damage is observed on the proof load survivors.
- Subsample group 4–R20 and T100. No damage is observed on the proof load survivors.
- Subsample 5–R40 and T100. The proof load survivors are damaged. The damage model is \([Y^*|X > 6.11] \overset{d}{=} [Y|X > 6.11] - 0.12/[Y|X > 6.11]\). The desired likelihood sample \( s \) from the distribution of \([Y|X > 6.11]\) is generated based on the observed sample \( s^* \) from the distribution of \([Y^*|X > 6.11]\).
- Subsample group 6–R60 and T100. No damage is observed on the proof load survivors.

In the cases of no damage (i.e. Subsample groups 1, 3, 4, and 6), the SPLD–with–shoulder approach is applied directly to obtain the MLE of \( \mathbf{\theta} \). In the cases of damage (i.e. Subsample groups 2 and 5), the augmented SPLD–with–shoulder approach is applied to obtain the MLE of \( \mathbf{\theta} \). Equivalently, the desired observation \( s \) for the likelihood function \( L_{PFY}(\mathbf{\theta}; s, y) \) is generated from the observed sample \( s^* \). Once the desired sample \( s \) is generated, the SPLD–with–shoulder approach is applied to obtain the MLE of \( \mathbf{\theta} \).

The MLE’s of \( \mathbf{\theta} \) based on Subsample groups 1 to 6 are reported in Table 7 with the corresponding standard errors. Table 7 also provides sample sizes for each sample. The MLE’s of \( \rho \) tell a consistent story that \( X \) and \( Y \) are highly correlated. The approximate 95% confidence intervals of \( \rho \) contain moderate to high correlation values. If there were a need to verify the strength properties, strength properties would not need both to be tested. We can collect \( X \) and predict \( Y \) from the collected
Given \( X = x \), we could predict \( Y \) by the conditional mean of \( Y \) given \( X = x \), i.e. \( \mu_y + \sigma_y \rho (x - \mu_x) / \sigma_x \). The estimated conditional mean of \( Y \) given \( X = x \) is

\[
\hat{\mu}_y + \hat{\sigma}_y \hat{\rho} (x - \hat{\mu}_x) / \hat{\sigma}_x,
\]

where \( \hat{\mu}_x, \hat{\sigma}_x, \hat{\mu}_y, \hat{\sigma}_y \) and \( \hat{\rho} \) are the MLE’s of \( \mu_x, \sigma_x, \mu_y, \sigma_y \) and \( \rho \), respectively.

### Table 7: Application: Summer–of–2011 experiment.

The MLE of \( \theta \) and its standard error are reported based on each SPLD–with–shoulder sample. The approximate 95% confidence intervals (CI) of \( \rho \) are also reported.

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#### 4.1 Summary

This section shows how to incorporate the damage model developed in Subsection 2.3 into the SPLD–with–shoulder approach described in Section 1. The augmented SPLD–with–shoulder design generates the desired sample \( s \) from \((Z, U)\) for the joint likelihood \( L_P{F_Y}(\theta; s, y) \) based on the observed sample \( s^* \) from \((Z^*, U)\) using the distributional relationship between \([Y|X > l_2]\) and \([Y^*|X > l_2]\). The parameter \( \theta \) is then estimated by the maximum likelihood method where the MLE of \( \theta \) maximizes the joint likelihood \( L_P{F_Y}(\theta; s, y) \) where \( s \) is generated from \( s^* \). Simulation studies indicate that the resulted MLE is an accurate estimate. The generalized SPLD–with–shoulder approach is illustrated with the Summer–of–2011 experiment data for assessing the dependence between \( MOR \) and the log–transformed \( UTS \). The high correlation found in the application has
important implications for lumber monitoring programs. If there is a need to assess MOR and UTS, the high correlation between MOR and the log–transformed UTS suggests that we do not need to measure both MOR and UTS. Instead, we reduce the cost of the lumber monitoring programs by collecting one of them and the other is predicted from the collected one.

5. CONCLUDING REMARKS

The research reported in this paper extends the approach in the celebrated paper of (Evans et al. 1984) that shows how statistics can in effect “break the same board twice”. The idea is that in step one a lumber specimen is strength tested by proof loading it in one failure mode to a specified level. If it fails the result is recorded. If it survives it is tested in a second strength testing until it fails and that result is recorded. Optimally the first test level should be set so that about 50% of the boards are broken–this maximizes the amount of information in the resulting sample about the correlation between the two different strengths. But it also damages the specimen so that the second measurement, where recorded, underestimates the strength in that mode. All results assume a bivariate normal distribution for the two strength properties so that the correlation measures the degree of dependence between them.

This paper has shown how to adjust for that damage and hence enable the correlation to be validly estimated. The result can then be used to construct a predictor of one strength from another, thus saving sampling costs in a long term monitoring program for the two strengths.

Simulation studies show that the method works–analytical study is ruled out due to the complexity of the method. Agreement on the estimated correlation using to independent samples where the preliminary and secondary tests are reversed further assure the validity of the method.

Although the sample used in the application seems large–around 900 specimens tested during the summer of 2016 in an expensive experiment, the sample was not representative of the entire lumber population of Canada. A follow up study involving such a sample would be desirable to ensure the practical relevance of the work. However, the resources needed for such a study would be appreciable.
6. ACKNOWLEDGEMENTS

We thank the whole group of investigators in the Forest Product Stochastic Modelling Group at FPInnovations, Simon Fraser University and the University of British Columbia, for their comments and work on the experiment referred to in this paper.

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