Duration of load modeling

Chun-Hao, Jim, and Samuel

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The dimension-corrected Canadian model:

\[ \frac{d}{dt} \alpha(t) \mu = [(a \tau_s)(\tau(t)/\tau_s - \sigma_0)_+]^b + [(c \tau_s)(\tau(t)/\tau_s - \sigma_0)_+]^n \alpha(t) \]

where

- \( 0 \leq \alpha(t) \leq 1 \) is the damage accumulated by time \( t \)
- \( \alpha(T_f) = 1 \) indicates that failure occurs at the (random) time \( T_f \)
- \( \tau_s \) (psi) = short term breaking strength
- \( \mu \) population mean time to failure
- \( a, b, c, n, \sigma_0 \) are piece-specific random effects that provide randomness in \( T_f \)
- \( \sigma_0 \) is the stress ratio threshold: damage starts to accumulate only when \( \frac{\tau(t)}{\tau_s} > \sigma_0 \).
Random effects

Modeling assumptions for each piece in the population:

\[ a | \mu_a, \sigma_a \sim N(\mu_a, \sigma_a) \]
\[ b | \mu_b, \sigma_b \sim \text{Log-Normal}(\mu_b, \sigma_b) \]
\[ c | \mu_c, \sigma_c \sim N(\mu_c, \sigma_c) \]
\[ n | \mu_n, \sigma_n \sim \text{Log-Normal}(\mu_n, \sigma_n) \]
\[ \sigma_0 | \mu_{\sigma_0}, \sigma_{\sigma_0} = \frac{\eta}{1 + \eta}, \eta | \mu_{\sigma_0}, \sigma_{\sigma_0} \sim \text{Log-Normal}(\mu_{\sigma_0}, \sigma_{\sigma_0}) \]

Parameter vector: \( \theta = (\mu_a, \sigma_a, \mu_b, \sigma_b, \mu_c, \sigma_c, \mu_n, \sigma_n, \mu_{\sigma_0}, \sigma_{\sigma_0}) \).

Given \( a, b, c, n, \sigma_0 \), the failure time can be computed by solving the ODE numerically.
Estimation

Goal: Estimate the ADM parameters based on a sample of observed failure times $y_1, \ldots, y_n$ under a constant-load test with level $\tau_c$.

The constant load test: $\tau(t) = kt$ for $t \leq T_0$, where $T_0 = \tau_c/k$, followed by $\tau(t) = \tau_c$ for $t > T_0$.

Problem: The likelihood $L(\theta|y_1, \ldots, y_n) = \prod_{i=1}^n p(y_i|\theta)$ is intractable since

$$p(y_i|\theta) = \int p(y_i|a, b, c, n, \sigma_0, \theta)p(a, b, c, n, \sigma_0|\theta) \, da \, db \, dc \, dn \, d\sigma_0$$

For a fixed value of $\theta$, $p(t_i|\theta)$ might be approximated by kernel-density estimates based on a large number of failure times simulated from the ADM. Not feasible in a parameter estimation routine where many $\theta$ proposals must be tried!
Estimation

Additionally, the test is truncated after a certain time (a few months to a few years).

Hence the observed data likelihood is in fact

\[
[P(Y > c|\theta)]^{n_c} \prod_{i=1}^{n-n_c} p(y_{obs,i}|\theta)
\]

where \( n_c \) is the number of pieces that survive until the end of the test, time \( c \).
Approximate Bayesian computation

To bypass the intractability of the likelihood, we adopt the approximate Bayesian computation (ABC) technique which only requires the ability to generate data from the model.

The key step in ABC is the approximation of the posterior

\[
\pi(\theta|y_{obs}) \approx \pi_{ABC}(\theta|s_{obs}) \propto \pi(\theta)p(s_{obs}|\theta)
\]

where \( s_{obs} = S(y_{obs}) \) for some summary statistic \( S(\cdot) \).

\[
p(s_{obs}|\theta) = \int \pi(y|\theta)K_\delta(S(y) - s_{obs})dy
\]

where \( K_\delta(\cdot) \) is a density kernel with bandwidth \( \delta > 0 \).

This permits a MCMC algorithm for sampling from the posterior of \( \theta \).
Approach is promising from simulated data. Plot of ‘brute-forced’ log-likelihood based on 500 ABC-MCMC draws.
ABC approach provides samples from posterior distributions of $\theta$

Simulate stochastic $\tau(t)$ that mimic live loadings

Together, we obtain a posterior distribution of $T_f$, the future time–to–failure, that accounts for uncertainty in the parameters and uncertainty in future loadings, which fulfills a key objective of ADMs.