Abstract

Tension proofloading is a quality control procedure designed to filter out weak and/or badly produced fingerjointed lumber. It is carried out under a stress level and rate (time to reach the stress level). Although proofloading removes weaker lumber pieces, it may also damage the pieces that survive proofloading. One objective of this study is to analyze the trade-off between the removal of weaker lumber and damage to the survivors. We model the tensile stress distribution of lumber pieces for a control sample that was not proofloaded and the distribution conditional on surviving proofloading for nine treatment groups. We find that both the 3-parameter Weibull and Log-Normal distributions are well suited for modelling the variation in stress in the samples of the study, whereas the 2-parameter Weibull is shown to be inadequate. Our analysis also includes modelling the probability of surviving proofloading as a function of stress level and rate. In combination with modelling the control and treatment stress distributions, it is shown that survivors are stochastically stronger than controls. For instance, survivors can have a tensile stress distribution with larger percentile values in the lower tail. The other objective of the paper is to optimize the proofloading conditions, that is, to theoretically model the tensile stress distribution of survivors under a range of possible proofloading conditions. Based on our analysis, survivors are stochastically strongest under a proofload rate of 25–30 seconds combined with a reasonably high stress level. The model shows that the suggested conditions do not destroy excessive amounts of lumber.
1 Introduction

Materials presented in Sections 1.1 and 1.2 are summarized from ? and ?.

1.1 What is Proofloading?

A fingerjointed piece of lumber consists of two pieces of lumber, each with built-in “fingers”, which are glued together to form a single piece of lumber. There are several quality problems that may arise during the manufacturing process, such as improperly made “fingers” and insufficient or nonexistent glue at the joints. Tension proofloading is a quality control method for testing structural fingerjointed lumber. During tension proofloading, a fingerjointed lumber is clamped at its two ends, and it is subjected to a pre-determined tension stress level and rate (time to reach target stress level). If the lumber breaks during proofloading, then it is recycled back into the production process, whereas those that pass are sent to market. Not only is proofloading a test on finger-joint quality, it is also a test on the quality of lumber pieces. However, the focus of this study is solely on the fingerjoint quality.

This report investigates two important issues regarding tension proofloading: (1) Whether proofloading is indeed beneficial, and (2) How to optimize the proofloading process? In order to discuss these two research objectives in better detail, we should first describe the data to be used for our analysis and the experiment that produced these data.

1.2 The Experiment and Data

In this experiment, the focus is on the fingerjoint itself, therefore the lumber pieces are all high quality specimens. There are 10 groups of lumber, each with 272 specimens. Furthermore, each group contains of 20% short-fingerjoints, the “finger” length being 3/8 inches to simulate defective lumber products. The other 80% consists of long-fingerjoints, with “finger” length set at 1 – 1/8 inches. Among the 10 groups of lumber, one group is referred to as “control”, where the lumber are not subjected to proofloading and tested to failure directly. Each lumber piece’s tensile strength (unit pounds) at the time of failure are recorded. Moreover, the tensile stress (unit pounds per squared inch or psi) is calculated for each recorded tensile strength. The data from this group would serve as a reference point for various analysis. The other 9 groups are the treatment groups, lumber from each treatment group is subjected to different combinations of proofload conditions listed below:

- Proofload stress levels at: 3800psi, 5000psi or 6100psi, which will be denoted as level A, B or C respectively.
- Using “s” to denote second(s), three rates at: 0.2s, 6s or 60s, which are rate 1, 2 or 3 respectively.
For example, B2 corresponds to samples with proofload set at load = 5000psi and rate = 6s, while A3 denotes samples with load = 3800psi and rate = 60s. These notations are used throughout this report. Within a treatment group, those that passed proofloading are referred to as “survivors” in this report, and these survivors are then tested to failure to record their tensile strengths/stress. To evenly match all lumber in the experiment, their moisture content and experimental conditions are carefully matched and monitored to remove exogenous factors.

1.3 Objectives of the study

As mentioned, there are two main objectives for the study:

1. Statistically analyze the benefit and disadvantages of tension proofloading. The benefit is that defective lumber (represented by short-fingerjoints) are discarded from a lumber population. However, depending on the proofload condition (load level and rate), surviving lumber may sustain damages.

The extent of proofload damages can be visualized using the sample data. Naturally, one can assume that the strongest lumber in a batch would survive proofloading. Let $n_{survivors}$ be the number of lumber pieces that survived proofloading under a treatment condition, by directly comparing the tensile stress of these survivors with the strongest $n_{survivors}$ lumber in the control sample, we may get a sense of damages inflicted by proofloading under this particular treatment condition. Figure 1 shows Q-Q plots of survivors’ tensile stress against those of the controls for each treatment, where each point corresponds to a set of tensile stress values of the same rank in the survivor and control sample. To interpret the Q-Q plot, use the diagonal line $x = y$ as a reference: the point lying under the line implies that for this survivor-control pair, survivor is weaker than the control, and vice versa for the point lying above the line. Also notice that a higher proofload stress level results in a smaller number of survivors (the corresponding sample size is displayed in the heading of each plot).

Table 1 shows the proportions of controls having greater tensile stress than the survivors of same rank for all 9 treatment groups. For instance, the proofloading condition A2 has proportion 0.702. To arrive at this number, first sort in decreasing order, the tensile stress values of the A2 survivors (sample size 265) and the controls, then calculate the proportion of the 265 survivor-control pairs where the control has greater tensile stress. Visually, these proportions are represented by the points lying under line $x = y$ in figure 1.

The results from Figure 1 and table 1 show the possibility of survivors sustaining damages through proofloading. Therefore an important question needs to be asked: is the deletion of weaker lumber enough to offset the damages suffered by the survivors, making
Table 1: Sample proportion of controls having greater tensile stress than the survivors of same rank.

<table>
<thead>
<tr>
<th>Stress level</th>
<th>Rate = 0.2s (1)</th>
<th>6s (2)</th>
<th>60s (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3800psi (A)</td>
<td>0.748</td>
<td>0.702</td>
<td>0.701</td>
</tr>
<tr>
<td>5000psi (B)</td>
<td>1.000</td>
<td>1.000</td>
<td>0.887</td>
</tr>
<tr>
<td>6100psi (C)</td>
<td>1.000</td>
<td>0.978</td>
<td>0.992</td>
</tr>
</tbody>
</table>

Figure 1: Q-Q-plots of ranked data: x-axis is the tensile stress of controls in 1000psi, while y-axis is the stress of survivors in same scale. The meanings of combinations A, B and C with 1, 2, and 3 are as explained in Section 1.2. n denotes the size of the surviving lumber for each treatment combination.

1. Proofload survivors a stronger lumber population than the original one?

2. Optimize the proofload process. We have sample data on survivors from 9 proofload conditions, meaning one would be able to estimate the distributions of 9 surviving lumber populations. The objective here is to further the analysis by building an estimation model that would predict the theoretical probability density function (pdf) of survivors from proofload conditions besides the ones in the experiment. This study should help us to better understand the combined effect of proofload stress level and rate.
1.4 Outline of the Report

To see if the survivors form a stronger lumber population than the controls, our approach is to model a pdf describing the tensile stress distribution of each lumber population. The stress distributions are then compared both visually and numerically, helping us to decide whether tensile stress proofloading results in a stronger lumber population. We further examine how pdf parameters and the proofload failure probability are affected by proofload settings, namely the aforementioned load \( l \), rate \( r \) and initial short-fingerjoint proportion (denoted as \( \delta \) later in the report). This analysis helps to optimize the proofloading.

Section 2 presents the data and preliminary analyses. Detailed modeling procedure for a single lumber population, its results and discussions are presented in Section 3. In Section 4, we implement a method of modeling mixture lumber population (a lumber population containing both short and long fingerjoints), followed by assessing any possible benefit of tension proofloading. The procedure, results and discussions on proofload optimization analysis are presented in section 5 ∼ 7. In Section 8, we will use the results from single-fingerjoint proofloading to make inferences on the effectiveness of tension proofloading on multi-fingerjoint ribbons. Section 9 is the conclusion.
2 Data on Homogeneous Lumber Populations

We use the term “homogeneous” to describe a lumber population that does not contain any subset of lumber with distinguishably different characteristics. As mentioned, our lumber samples contain short and long fingerjoints, where data in a “homogeneous” sample would be the tensile stress data for either fingerjoint type from each of the 10 lumber groups (control and 9 treatments), so we have data on 20 homogeneous batches of lumber.

In this report, all analyses are conducted using the data on tensile stress, which is a measurement of lumber strength per unit cross-sectional area. Tensile stress is measured in pounds-force per square inch and the unit symbol is psi. Table 2 and figure 2 present the summary statistics and histograms for both short and long fingerjoints from the control sample, while figure 3 shows the histograms of long fingerjoints from the 9 survivor samples.

From the histograms, we can see that a “lower tail distribution” exists for the long-fingerjointed control as well as each survivor at level A. Such a lower tail becomes much less pronounced, or disappears when one increases the proofload stress level. Moreover, we notice that the survivor sample size decreases as the stress level increases.

Table 2: Summary statistics for both short and long fingerjoints of the control sample.

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>1st Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>3rd Quartile</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>short</td>
<td>3.41</td>
<td>3.94</td>
<td>4.16</td>
<td>4.26</td>
<td>4.56</td>
<td>5.37</td>
</tr>
<tr>
<td>long</td>
<td>3.32</td>
<td>6.01</td>
<td>6.61</td>
<td>6.52</td>
<td>7.17</td>
<td>9.06</td>
</tr>
</tbody>
</table>

Figure 2: Sample data histogram for both short and long fingerjoints of the control sample. The x-axis is the sample tensile stress in 1000psi. n is the size of the lumber sample in each histogram.
Figure 3: Sample data histogram of long fingerjoints from the 9 survivor groups. The x-axis is the sample tensile stress in 1000psi. \( n \) is the size of the long fingerjointed survivors for the respective treatment combination.

3 Modelling the Tensile Stress Distribution of a Homogeneous Population

Our strategy is to use appropriate pdf to model the tensile stress distribution of a homogeneous lumber population. This section contains five subsections. The first one introduces the 3 candidate distributions for modelling the lumber tensile stress distribution: 2 or 3 parameter Weibull and log-normal distribution. The second subsection discusses the way to estimate, or “fit”, these distributions using our lumber data. The following two subsections present the fitted distributions and evaluate the quality of these fits. One subsection does so for the control populations, and the other for the proofloading survivors. The last subsection summarizes the important findings.
### 3.1 Candidate Probability Density Functions

Throughout this report, tensile stress is denoted by the random variable $X$. As discussed by ?, common probability density functions used for modelling lumber stress are Weibull and Log-Normal distributions.

The Weibull comes in either 2-parameter or 3-parameter form. The 3-parameter Weibull distribution has pdf

$$f_X(x|\alpha, \beta, c) = \frac{\beta}{\alpha} \left( \frac{x-c}{\alpha} \right)^{\beta-1} \exp\left( -\left( \frac{x-c}{\alpha} \right)^\beta \right), \quad x \geq c, \alpha, \beta > 0$$  \hspace{1cm} (1)

where $c \leq x$ is referred to as the location parameter, $\alpha$ and $\beta$ are scale and shape parameters. $\alpha$, $\beta$ and $c$ are parameters to be estimated. When $c$ is a known parameter, one arrives at a 2-parameter Weibull distribution. Since $c$ is not known for our study, we specify $c = 0$ for our 2-parameter Weibull. It implies that random variable $X$ can be arbitrarily close to 0.

Log-Normal distribution has pdf

$$f_X(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left( -\frac{(\log(x) - \mu)^2}{2\sigma^2} \right), \quad x > 0, \quad -\infty < \mu < \infty \text{ and } \sigma > 0.$$  \hspace{1cm} (2)

A property of log-normal distribution is that if $X \sim \log-N(\mu, \sigma)$, then $\log(X) \sim N(\mu, \sigma)$. We would make use of such convenience in later analysis.

By estimating the parameters in $f_X(x)$ using a sample of lumber data, we would obtain an estimated stress distribution for the lumber population from which this data is sampled. A properly estimated pdf would provide valuable insights when comparing strength between lumber populations. In all, we can estimate density functions (1) and (2) for 20 lumber populations.

However, for a proofloaded population (survivors), there exists an alternative modelling method. The underlying philosophy of this alternative method is that since one may regard survivors as the strongest lumber in a proofloaded batch, their tensile stress would therefore be modelled by a conditional distribution. If $f_X(x)$ is the stress distribution for the lumber prior to proofloading, then the survivor distribution is described by function

$$f_X(x|x > l) = \frac{f_X(x)}{\tilde{F}_X(l)}, \quad x > l.$$  \hspace{1cm} (3)

Here, $l$ is the proofload stress level and $\tilde{F}_X(l)$ is the probability that $X$ is greater than $l$.

The density curve of such conditional distribution has a “clear-cut” lower bound, whereas the density of Weibull or log-normal distribution has a “tapering” lower tail. If we were to utilize the first (and simpler) modelling technique, we are implying the existence of damages amongst survivors, machine/measurement errors and other factors that contribute to this tapering of...
lower tail. This alternative method seems to be a reasonable modelling approach, especially considering the lack of lower tail distribution displayed by some of the survivors (as seen in Figure 2 and 3). However, this alternative method delivered less than ideal results and its analysis is presented in Appendix 2 instead.

3.2 Estimating the Parameters of a Distribution

As mentioned, a distribution is estimated by estimating the parameters in the distribution. In this report, all parameter estimates are Maximum Likelihood Estimators (MLEs). In formal statistical definition, likelihood has expression \( f(x_1, \ldots, x_n | \theta) \equiv \prod_{i=1}^{n} f(x_i | \theta) \), it is a function of unknown parameters \( \theta \) given known data \( x \). In our case, \( \theta \) can consist of Weibull parameters \( \{\alpha, \beta, c\} \) or log-normal parameters \( \{\mu, \sigma\} \). By finding the values of \( \theta \) that maximize the likelihood (MLEs), one would obtain an estimated distribution that makes our observed data \( x \) “most likely”. In non-statistical terms, the MLE produces an estimated distribution that “best describes” the shape (histogram) of the data.

The MLE can be found analytically for the log-normal while it is obtained by means of a Newton-type algorithm for both the 2 parameter (2-p) and 3 parameter (3-p) Weibull. In statistics packages R,\(^1\) there are a number of functions that can calculate the MLE for the log-normal distribution. The one we used is called \texttt{fitdistr()}\(^\text{2}\), where the required input is simply a vector of sample data. The output includes the MLE, standard error of MLE, maximized log-likelihood value, etc.

We are not aware of any function that finds the MLE for a 3-p Weibull, so we developed a code for that propose (see Appendix 1). We first wrote a function having data and parameter vector as inputs, and the negative log-likelihood as output. By writing this function as such, \texttt{R} function \texttt{nlm()} could be used to find a set of parameters that minimize the negative of the log-likelihood for a given dataset, i.e., for finding the MLE’s. The minimizing algorithm in \texttt{nlm()} is defaulted as Quasi-Newton method, which is the one we used. Furthermore, \texttt{nlm()} can be specified to deliver the resultant standard errors for the MLE’s (\texttt{?}). We also wrote a modified version of this function in which the parameter \( c \) is fixed to a user defined value, so setting \( c = 0 \) would allow me to obtain 2-p Weibull MLE’s. It should be noted that the aforementioned \texttt{fitdistr} also finds the MLE for the 2-p Weibull, since results from our own algorithm corresponded very closely with those from \texttt{fitdistr}, we used our own code instead.

Lastly, the tensile stress values were scaled to \( 1/100 \)th to aid convergence during the negative log-likelihood minimization. All statistical analyses presented in this report are implemented with data scaled to \( 1/100 \)th.

\(^{1}\text{R is a computer language that is geared toward statistical related computing and analysis. It is widely used amongst statisticians.}\)
### 3.3 Non-proofloaded Lumber (Control)

#### Results for 2 and 3 Parameter Weibull

Table 3 shows the MLEs and maximized log-likelihood. In this paper, the “hat” notation above a parameter denotes an estimate. The parameters $\alpha$, $\beta$ and $c$ correspond to the Weibull distribution in (1).

<table>
<thead>
<tr>
<th></th>
<th>short fingerjoint</th>
<th>long fingerjoint</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\beta}$</td>
<td>$\hat{\alpha}$</td>
</tr>
<tr>
<td>3-parameter</td>
<td>2.010</td>
<td>1.045</td>
</tr>
</tbody>
</table>

The 3-parameter Weibull delivered a higher likelihood as expected. This is because that 2-parameter model ($c = 0$) is a subset of 3-parameter model and it maximizes the likelihood under only 2 parameters instead of 3.

One may apply the likelihood ratio test to see if two models, with 2-parameter Weibull being a nested model of the 3-parameter one, are significantly different in terms of log-likelihood. Let $D$ be 2 times the difference of log-likelihoods, so $D_{\text{short}} = 2 \times (-34.552 - (-40.865)) = 12.626$ and $D_{\text{long}} = 2.108$ similarly. $D$ is approximately Chi-squared distributed with 1 degree of freedom, and the p-value for this likelihood ratio test is 1 minus the cumulative distribution function (cdf) of $D$ under said Chi-squared distribution. Our decision rule is that if p-value $\leq 0.05$, the two log-likelihoods are determined to be significantly different. The p-value are $\approx 0$ and 0.1465 for short and long fingerjoints respectively. So for the short fingerjoints, 3-p Weibull is a significant improvement over the 2-p, whereas their difference is not statistically significant for the long fingerjoints.

Table 4 shows the results from the Chi-squared test. This is a test that measures the goodness-of-fit of our estimated distributions. It is carried out as follow:

1. Data is separated into $k$ bins.
2. The observed and expected frequencies $O_i$ and $E_i$, $i = 1, \ldots, k$, are obtained for each bin. $E_i$ is calculated as $(F(x_{i+1}) - F(x_i)) \cdot N$, where $F(x)$ is the cdf of a fitted Weibull, $\{x_{i+1}, x_i\}$ is the range for each bin $i$ and $N$ is the total sample size.
3. Calculate the goodness-of-fit statistic $G^2 = \sum_{i=1}^{k}(O_i - E_i)^2 / E_i$.
4. The p-value of this test is 1 minus the cdf of $G^2$ under the Chi-squared distribution with degrees of freedom $k - p - 1$, $p$ being the number of parameters.
The null hypothesis states that our estimated distribution is consistent with the observed data, and we would reject this null hypothesis if p-value ≤ 0.05. As shown in table 4, the 3-parameter Weibull delivered p-values noticeably larger than 0.05 for both fingerjoints. Thus, we failed to reject the null hypothesis. However, failing to reject the null hypothesis does not lead to the conclusion that the null is correct. It means that, given our data, we do not have the evidence to reject 3-parameter Weibull as an useful distribution. An equivalent statement is: the fitted 3-p Weibull described the data well enough that the Chi-squared test could not reject it. Although the 2-parameter Weibull was not rejected for long fingerjoints, the fit for short fingerjoints barely passed the Chi-squared test. More detailed test results are shown in Appendix 3.

Table 4: Chi-squared test p-value for Weibull fits presented in table 3.

<table>
<thead>
<tr>
<th></th>
<th>3-p short</th>
<th>2-p short</th>
<th>3-p long</th>
<th>2-p long</th>
</tr>
</thead>
<tbody>
<tr>
<td>p-value</td>
<td>0.253</td>
<td>0.070</td>
<td>0.549</td>
<td>0.568</td>
</tr>
</tbody>
</table>

It deserves mention that one may apply the constraint \( c_{\text{long}} \geq c_{\text{short}} \) when fitting the 3-p Weibull. As mentioned in section 2, the 3-p Weibull has support \( X \geq c \). Since it stands to reason that long-fingerjointed lumber belong to a “stronger population” than short fingerjoints, we would expect \( c_{\text{long}} \geq c_{\text{short}} \). However, for certain samples such as the control, we have a defective piece of lumber amongst long fingerjoints in which the lumber broke at a location away from the actual fingerjoint, resulting in a weaker tensile stress than the weakest of short fingerjoints. This resulted in \( \hat{c}_{\text{long}} < \hat{c}_{\text{short}} \) as shown in table 3. This can be adjusted by adding the constraint \( \hat{c}_{\text{long}} \geq \hat{c}_{\text{short}} \): when \( \hat{c}_{\text{long}} < \hat{c}_{\text{short}} \), we would fix \( c_{\text{short}} \) as \( \hat{c}_{\text{long}} \) and estimate the other two parameters for \( f_{\text{short}}(x) \). So mathematically speaking, we are simply fitting a 2-p Weibull, but instead of \( c_{\text{short}} = 0 \), we now have \( c_{\text{short}} = \hat{c}_{\text{long}} \). The log-likelihood for the long fingerjoints under this constrained fit is \(-38.883\). Using similar likelihood ratio test as the one before, we have \( D_{\text{short}} = 2 \times (-34.552 - (-38.883)) = 8.662 \) and p-value of 0.003, i.e., the log-likelihood is significantly smaller when fixing the location parameter at \( c_{\text{short}} = 1.948 \) compared to 3-p Weibull fit. The result from the Chi-squared test is noticeably worse as well. The decision on whether to apply this type of constraint depends on a researcher’s prior experience and belief regarding the lumber and associated data. We did not implement such a method in this report.

Log-Normal Distribution and Comparison with 3-p Weibull

We now focus on log-normal distributions and comparisons with the 3-p Weibull. Table 5 shows the parameter estimates for 3-p Weibull and log-normal distributions, and figure 4 shows their
QQ-plot.

On the QQ-plot, the y-axis is the ranked sample data of size \( n \). For each rank \( k = 1, ..., n \), one may calculate the sample cdf using \( (2 \cdot k - 1)/n \) (in \( \mathbb{R} \), the cdf is calculated in such manner for \( n > 10 \)). The x-axis is the quantile of the corresponding cdf of our estimated distribution. A good alignment of \( \{x, y\} \) points onto the line \( y = x \) implies that our model corresponds well with the actual data. To add another means of visualizing the goodness-of-fit, we plot the estimated 3-p Weibull and log-normal distribution over the data histogram, as shown in figure 5. A well estimated distribution should closely resemble the shape of the data histogram.

Visually examine the plots, it is obvious that 3-p Weibull is a good model for both types of fingerjoints and log-normal is suitable for short fingerjoints. However, deficiencies arise with the log-normal distribution when modelling long fingerjoints. If points in a QQ-plot lie below the line \( x = y \), it means the quantile of fitted distribution is larger than corresponding sample quantile, i.e., fitted distribution over-estimates the sample data. This is visually evident for the lower tail of fitted log-normal distribution on long-fingerjointed controls. The issue is reiterated in figure 5, where the fitted log-normal distribution displays a thinner lower tail than both the sample data and 3-p Weibull, which shows that the log-normal would over-estimate lower quantiles. Furthermore, we obtained a rather small Chi-squared test p-value of 0.074, which is rather close to rejecting the hypothesis of “acceptable fit”.

Table 5: MLE values and maximized log-likelihood under 3-p Weibull and log-Normal fit.

<table>
<thead>
<tr>
<th></th>
<th>3-p Weibull</th>
<th>Log-normal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{\beta} )</td>
<td>( \hat{\alpha} )</td>
</tr>
<tr>
<td>short fingerjoint</td>
<td>2.010</td>
<td>1.045</td>
</tr>
<tr>
<td>long fingerjoint</td>
<td>5.164</td>
<td>4.969</td>
</tr>
</tbody>
</table>

Table 6: Chisq-test p-values for 3-p Weibull and log-normal fit.

<table>
<thead>
<tr>
<th></th>
<th>Weibull short</th>
<th>log-normal short</th>
<th>Weibull long</th>
<th>log-normal long</th>
</tr>
</thead>
<tbody>
<tr>
<td>p-value</td>
<td>0.253</td>
<td>0.166</td>
<td>0.549</td>
<td>0.074</td>
</tr>
</tbody>
</table>

Conclusion for the Controls

Based on results from the control sample, the 3-parameter Weibull (\( \gamma \neq 0 \)) seems the most sensible choice. It gives noticeable improvement over the 2-parameter Weibull when modelling short-fingerjoints. Moreover, one should expect the high grade lumber to break at some stress
Figure 4: QQ-plots of theoretical quantiles (calculated using fitted distributions) against sample quantiles. Stress values are shown in 1000 psi. “Long” and “short” in the plot heading represent short fingerjoint and long fingerjoint respectively.

Figure 5: Fitted 3-p Weibull (solid line) and Log-Normal (dashed line) distribution over the histogram of control data, $x$-axis is the sample tensile stress shown in 1000 psi, $y$-axis is the density.

load higher than 0. Compared to log-normal, the 3-parameter Weibull did a better job of modelling the lower tail distribution of long fingerjoints. The usefulness and appropriateness of the log-normal and 2-p Weibull will come under further investigation in following subsections.
3.4 Proofloaded Lumber (Survivors)

One may simply model the survivor tensile stress using either of the two Weibull distributions or the log-normal distribution just as for the control population.

Figures 6 shows the plot of theoretical quantiles of fitted distribution vs. sample quantiles (QQ-plot per Figure 4) for long-fingerjointed survivors of level A. Figures 7 and 8 show the same type of QQ-plots for long-fingerjointed survivors of Level B and C. The most noticeable result is the lack-of-fit exhibited by the 2-parameter Weibull. The lower tail points in the QQ-plot lies above the $y = x$ line by significant margins, especially for level B and C survivors. This implies that the fitted 2-p Weibull would seriously underestimate the lower quantiles of the lumber stress distribution. Such an outcome is not surprising considering that 2-p Weibull specifies the minimum possible tensile stress as 0 (remember that our 2-p Weibull is specified as $c = 0$), which is unlikely for survivors. Indeed, the minimum tensile stress of our survivor samples is quite high. Intuitively, a 2-p Weibull model with location $c = 0$ would distribute probability unnecessarily on the lower stress region, thus underestimating the lower quantiles. This result, along with the relative lack-of-fit for the control data, led us to dismiss the 2-parameter Weibull as a candidate distribution.

For level-A survivors, the log-normal displayed a slight lack-of-fit on the upper tail for all 3 proofload rates, while the fit quality is good otherwise. The 3-p Weibull delivered a consistently good fit for all 3 groups of level-A survivors. This could be explained by the fact that at proofload level A, only 6 pieces of lumber failed proofloading (3 for both types of fingerjoints) under A1, 7 pieces (6 short and 1 long) under A2 and 8 pieces (all short) under A3. Therefore the survivor sample from level A should closely resemble the characteristics of the controls, for which unlike the 3-p Weibull, the log-normal has shown difficulties in modelling the tail distribution.

On the other hand, both the log-normal and the 3-p Weibull showed a similar goodness-of-fit for survivors of level B and C. Except for shared deficiencies in modelling a few points on the upper quantiles, our estimated 3-p Weibull and log-normal distribution modelled the tensile stress distributions of level B and C survivors satisfactorily.

3.5 Summary and Discussion

We first conclude that the 2-parameter Weibull is inadequate for modelling the tensile stress of proofloading survivors. The more complex 3-parameter Weibull distribution delivered better fits for all our data, and proved to be a recommendable pdf for the purpose of modelling tensile stress distributions of a homogeneous lumber population. However, more nuanced decision making is required when assessing the merit of the log-normal distribution.

The proceeding analyses has shown that 3-parameter Weibull is indeed preferable to log-
normal when modelling the control and survivors of low-stress proofloading. On the other hand, the log-normal delivered nearly identical results to the 3-p Weibull when modelling the tensile stress of survivors from a more rigorous proofloading condition, such as load level B or C.

In our opinion, deficiency of the log-normal can be attributed by the nature of both the distribution itself and our data. If $Y \sim \log - N(\mu, \sigma)$, then the MLEs are $\hat{\mu} = [\sum \log(y)]/n$ and $\hat{\sigma} = [\sum (\log(y) - \hat{\mu})^2]/n$, where $y$ is the sample values and $n$ the sample size. In plain language, the MLEs of log-normal parameters are the sample mean and sample standard deviation. The values in our long fingerjointed data resulted in $\hat{\mu}$’s and $\hat{\sigma}$’s that gave distribution curves a near-symmetric or left-skewed shape. However, as the histograms of long fingerjointed control and A-level survivors show, the data are skewed to the right instead. The discrepancies between the MLEs and “skewness” of our data explain the lack-of-fit exhibited by log-normal distribution for control and survivors of low-stress proofloading. Figure 9 shows the fitted 3-p Weibull and log-normal over the data histogram for the long-fingerjointed survivors. Notice that some fitted log-normals are skewed to the left at instances where the data histogram and fitted 3-p Weibull are not.

Nevertheless, the log-normal distribution delivered good fits in most situations, and given the results in ?, there is reason to believe that the lack-of-fit derives in part by our particu-
Figure 7: QQ-plot of fitted 3-p Weibull, log-normal and 2-p Weibull for B-long (long fingerjointed survivors from proofload level B), tensile stress values are shown in 1000psi.

lar dataset. More importantly, the application of the log-normal distribution gives a ready-interpretible theoretical justification (statistically) for the analyses in Section 5 and onward. In the end, we decided to keep log-normal as a viable alternative to the 3-parameter Weibull.
Figure 8: QQ-plot of fitted 3-p Weibull, log-normal and 2-p Weibull for C-long (long fingerjointed survivors from proofload level C). The tensile stress values are shown in 1000psi.
Figure 9: Fitted 3-p Weibull (red) and Log-Normal (green) distribution over histogram for the long-fingerjonted survivors. $x$ – axis is the sample tensile stress shown in 1000psi, $y$ – axis is the density.
In the previous section, we explored ways to model a homogeneous population of lumber. In this section, we use the results from last section as a basis for the creation of means of modelling a mixture of lumber populations, namely a population containing both short and long fingerjoints.

### 4.1 pdf for mixed populations

A mixed lumber population is not only true for our data, in real manufacturing settings, one must take into account the existence of defects or more generally, subpopulations of lumber within a homogeneous production batch. Mixture distribution is applied to model the tensile stress of a mixed lumber population. It has form

\[ f_X(x) = p \cdot f_{\text{short}}(x) + (1 - p) \cdot f_{\text{long}}(x), \quad (3) \]

where \( p \) is the proportion of short fingerjoints or defects, \( f_{\text{short}}(x) \) and \( f_{\text{long}}(x) \) are the pdfs of short and long fingerjoints. In a nutshell, the random variable \( X \) represents the tensile stress of a mixed population, the distribution for the values of \( X \) being expressed by the two pdf’s weighted according to the parameter \( p \).

When \( f_X(x) \) is a 3-parameter Weibull, the range of \( X \) is the minimum of the two location parameters in \( f_{\text{short}}(x) \) and \( f_{\text{long}}(x) \), i.e, \( x \geq \min(c_{\text{short}}, c_{\text{long}}) \). However, because we fit \( f_{\text{short}}(x) \) and \( f_{\text{long}}(x) \) separately using short and long fingerjointed data, it is entirely possible that a part of the mixture data \( x \) could be smaller than either \( c_{\text{long}} \) or \( c_{\text{short}} \). Therefore when mixing the distribution for 3-p Weibull, one should keep in mind that \( f_{\text{short}}(x) \) and \( f_{\text{long}}(x) \) only have values in the data range \( x \geq c_{\text{short}} \) and \( x \geq c_{\text{long}} \) respectively. Outside of these range, \( f_{\text{short}}(x) \) or \( f_{\text{long}}(x) \) takes the value 0 in the mixture distribution.

### 4.2 The Equation of \( p \)

We have already estimated \( f_{\text{short}}(x) \) and \( f_{\text{long}}(x) \) for the control and 9 survivor groups, the remaining piece of the puzzle generated by equation (3) is to find proportion parameter \( p \).

For lumber without the proofload treatment, the mixing proportion \( p \) may be estimated using relevant data from the lumber mill. In our case, such a population is sampled as the control group, and \( \hat{p} \) is 20% or 0.2 based on the design of our experiment.

As for the proofloading survivors, \( p \) would generally have form

\[ p(\delta, l, r) = \frac{\delta \cdot (1 - \pi_s(l, r))}{\delta \cdot (1 - \pi_s(l, r)) + (1 - \delta) \cdot (1 - \pi_l(l, r))}. \quad (4) \]
Here, $\delta$ is the proportion of short fingerjoints prior to proofloading (also 0.2 in our data), $\pi_s$ and $\pi_l$ are failure probabilities for short and long fingerjoint accordingly. Moreover, $\pi$ can be modeled with proofload level ($l$) and rate ($r$), which makes proportion parameter $p$ a function of $\{\delta, l, r\}$. Later on, we show the way of modelling $\pi$, and in turn $p$, as a function $l$ and $r$. For this section, we use the sample proportion of short-fingerjoints in the survivor data as an estimate for $p$ and construct empirical mixture distributions via function (3).

### 4.3 Comparisons of the Proofload Survivors and the Control Distribution

The tensile stress of control and survivor populations can be compared in a number of ways using the mixture distributions we built. For example, one may calculate $P(X < l)$: the probability of a random piece of lumber from certain population being weaker than some load level $l$. Table 7 shows such probabilities based on 3-p Weibull mixture model, where values in each cell $ij$ correspond to probability $P(X_{ij} < l_i)$. For instance, cell B2 corresponds to the probability of survivors from proofload condition B2 ($l_B = 5000\text{psi}$, $r_2 = 6s$) being weaker than $l = 5000\text{psi}$, while the column “control” shows $P(\text{control} < l_i)$ for each row $i$. These probabilities are cumulative density functions (cdfs). They can be calculated by first, derive the expression of the 3-p Weibull cdf, then plug in the parameter estimates and $l$ values to find the probabilities.

For every row in table 7, the probabilities in column 1–3 are smaller than column 4 probability of the same row. These results indicate that the probability of a randomly chosen lumber breaking at a given stress level is smaller for the proofloaded population. In other words, the lower quantiles are higher for the proofloaded lumber.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>control</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.015</td>
<td>0.011</td>
<td>0.011</td>
<td>0.043</td>
</tr>
<tr>
<td>B</td>
<td>0.053</td>
<td>0.065</td>
<td>0.093</td>
<td>0.248</td>
</tr>
<tr>
<td>C</td>
<td>0.264</td>
<td>0.148</td>
<td>0.339</td>
<td>0.461</td>
</tr>
</tbody>
</table>

Table 7: Probability of failing below proofload level calculated using fitted 3-p Weibull mixture distribution.

Another means of comparison is to visually compare the estimated distributions between control and survivor populations. Figure 10 plots the mixture distributions of the 9 survivor groups superimposed with the controls, where once again, the mixture distribution is constructed via eq. (3), with $p$ being the sample proportion of short fingerjoints and individual $f_X(x)$ being 3-p Weibull. Proofloading discarded most short fingerjoints under level B and C.
Figure 10: Estimated survivor population (red line) vs. control (blue line) under 3-p Weibull mixture distribution.

Proportions of short fingerjoints were around 5% for both B1 and B2, and the estimated mixture distributions are visually unimodal. As for level C, all short fingerjoints were destroyed, making each survivor distribution simply a 3-p Weibull. As the plots show, all 9 survivors’ lower-tail distributions lie at higher stress region than that of the control, i.e., higher low-quantile values for the survivors. Furthermore, survivors are more heavily distributed at the high stress region between 7000psi and 9000psi, meaning it is more probable to randomly receive a high-quality piece of lumber from a proofloaded population.

It is true that an unfortunate side effect of tension proofloading is that survivors could sustain damage through the process. However, as our results shown, the deletion of weak lumber (production defects in reality) offsets this side effect. In the end, tension proofloading would create a stronger fingerjointed lumber population than those did not receive the treatment.
5 Modeling the Survivor Tensile Stress as a Function of Load and Rate of Loading

In the previous section, we estimated stress distributions for the survivors from 9 combinations of proofload stress levels and rates. The shortcoming of this modelling method is that only lumber populations with available data can be analyzed. Therefore, the follow-up question would be whether we can establish a model that estimates distributions for survivors of any proofload condition. In the previous section, we mentioned that the mixture proportion $p$ can be the function of load ($l$), rate ($r$) and pre-proofload proportion $\delta$. Using log-normal as the distribution of choice for the survivors, we may further consider parameters $\mu$ and $\sigma$ as the function of $l$ and $r$.

In this and the following sections, we would develop regression models for $p$, $\mu$ and $\sigma$ based on variables $\{l, r, \delta\}$, then use the results from the regression analysis to estimate the mixture distribution of survivors from any “reasonable” combinations of $l$ and $r$. By “reasonable”, we simply mean that when choosing $l$ and $r$ values to model the distribution, they should be chosen within the range of our experimental values, not ones far removed from them.

The results from Section 3 indicates that the 3-parameter Weibull might be a better distribution to build an estimation model with. To our knowledge however, there are no established statistical theories in which to derive regression models for the 3-p Weibull parameters $\alpha$, $\beta$ and $c$. The development of a 3-p Weibull based regression model is mentioned as a possible future research topics in our conclusion section. In this report, we develop the aforementioned regression models for parameters of log-normal distribution.

5.1 Log-Normal Parameter $\mu(load, rate)$

To model log-normal parameter $\mu$ as a function of $l$ and $r$, we would make use of two convenient properties:

- If $X \sim \log - N(\mu, \sigma)$, then $\log(X) \sim N(\mu, \sigma)$.

- In a typical linear regression where response is $Y(c) \sim N(\mu(c), \sigma(c))$, covariates being $\{c_1, \ldots, c_p\}$ and independent random error having mean 0, $\mu(c)$ can be defined by function $\mu(c) = c_1 \cdot \beta_1 +, \ldots, c_p \cdot \beta_p$ where $\beta$’s are the coefficients to the covariates.

Therefore by building a regression model with log(tensile stress) as the response, and functions of $\{l, r\}$ as the covariates, we would obtain an estimating function for $\mu$ under any values of $l$ and $r$. 
For said regression of $\log$(tensile stress) on load and rate, one can start with full second-order order model

$$\log(\text{tensile stress}) = \beta_0 + \beta_1 \cdot l + \beta_2 \cdot r + \beta_3 \cdot l^2 + \beta_4 \cdot r^2 + \beta_5 \cdot l \cdot r + \varepsilon$$ \hspace{1cm} (5)

where $l$ is the proofload level, $r$ is the rate and independent random error $\varepsilon \sim N(0, \sigma^2)$ ($\sigma^2$ defines the variance of random error, not to be confused with our log-normal parameter $\sigma$). We then used backward elimination with Akaike Information Criterion (AIC) as the reference to narrow down our model. The rule here is that a regression model is the “best” when it has the smallest AIC value. This backward elimination procedure can be implemented by either `stepAIC` or `step` function in R. We found that for long fingerjoints, the full model delivered the best result, while short fingerjoints has the linear form

$$\log(\text{stress}) = \beta_0 + \beta_1 \cdot l + \beta_2 \cdot r + \beta_3 \cdot r^2 + \varepsilon.$$ \hspace{1cm} (6)

The covariates in these regression models are checked for significance, and the regression models are subjected to diagnostic tests on residuals and influential outliers. Tables 8 and 9 shows the numerical output from the regression analysis, where the values under column “Estimate” are the regression coefficients.

Table 8: Coefficient estimations, standard errors and significance test values for the long fingerjoints, or equation (5)

|          | Estimate | Std. Error | t value | Pr(>|t|) |
|----------|----------|------------|---------|----------|
| Intercept| 2.3506   | 0.1227     | 19.16   | 0.0000   |
| load     | -0.2420  | 0.0513     | -4.72   | 0.0000   |
| rate     | 0.0052   | 0.0017     | 3.12    | 0.0018   |
| load$^2$ | 0.0273   | 0.0052     | 5.22    | 0.0000   |
| rate$^2$ | -0.0001  | 0.0000     | -2.18   | 0.0297   |
| load\cdot rate | -0.0004 | 0.0001 | -3.00 | 0.0028 |

5.2 Log-Normal Parameters $\sigma(\text{load}, \text{rate})$

To estimate $\sigma$ under different load and rate values, we would first log transform $\sigma$ to give it the range $(-\infty, \infty)$ instead of $\sigma > 0$. The full second-order linear model has the form

$$\log(\sigma_i) = \gamma_0 + \gamma_1 \cdot l_i + \gamma_2 \cdot r_i + \gamma_3 \cdot l_i^2 + \gamma_4 \cdot r_i^2 + \gamma_5 \cdot l_i \cdot r_i + \varepsilon,$$ \hspace{1cm} (7)
Table 9: Coefficient estimations, standard errors and significance test values for the short fingerjoints, or equation (6)

|                | Estimate | Std. Error | t value | Pr(>|t|) |
|----------------|----------|------------|---------|----------|
| (Intercept)    | 1.2063   | 0.0570     | 21.16   | 0.0000   |
| load           | 0.0850   | 0.0138     | 6.15    | 0.0000   |
| rate           | -0.0089  | 0.0034     | -2.60   | 0.0101   |
| rate²          | 0.0001   | 0.0001     | 2.58    | 0.0105   |

where \( i = 1, \ldots, 9 \) for long fingerjoints (denoting 9 treatment groups) and \( i = 1, \ldots, 6 \) for short fingerjoints (only 6 treatment groups have short-fingerjointed survivors). The sample data for \( \log(\sigma) \) are simply the parameter estimates \( \log(\hat{\sigma}) \) obtained previously.

Table 10 shows the AIC values under combinations of covariates for the regression fits on long-fingerjointed data. The regression model with just load and rate as covariates has the smallest AIC value, making it the best choice for modelling \( \log(\sigma) \). Although the covariate rate was found to be insignificant with p-value = 0.1972 (table 11), it was decided not to discard it as we thereby avoid model over-simplification.

Table 10: AIC values from backward elimination of equation (7) for long fingerjoints.

<table>
<thead>
<tr>
<th></th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>full model as in equation 7</td>
<td>-21.66</td>
</tr>
<tr>
<td>reduced to covariates ( {l, r, r^2, l*r} )</td>
<td>-23.61</td>
</tr>
<tr>
<td>reduced to covariates ( {l, r, l*r} )</td>
<td>-24.47</td>
</tr>
<tr>
<td>reduced model with covariates ( l ) and ( r )</td>
<td>-24.68</td>
</tr>
</tbody>
</table>

Table 11: Key results for the \( \log(\sigma) \)-estimating regression model for long fingerjoints.

|                | Estimate | Std. Error | t value | Pr(>|t|) |
|----------------|----------|------------|---------|----------|
| (Intercept)    | -1.2796  | 0.0875     | -14.63  | 0.0000   |
| load           | -0.1695  | 0.0171     | -9.90   | 0.0001   |
| rate           | 0.0009   | 0.0006     | 1.45    | 0.1972   |

Using the same procedure as above, we arrive at regression model with covariates \( \{l, r, l*r\} \) for short fingerjoints, and the results shown in table 12.
Table 12: Key results for the log(\(\sigma\))-estimating regression model for short fingerjoints.

|                  | Estimate | Std. Error | t value | Pr(>|t|) |
|------------------|----------|------------|---------|----------|
| (Intercept)      | -1.2288  | 0.3187     | -3.86   | 0.0611   |
| load             | -0.2834  | 0.0718     | -3.95   | 0.0585   |
| rate             | -0.0345  | 0.0092     | -3.77   | 0.0637   |
| load-rate        | 0.0086   | 0.0021     | 4.15    | 0.0535   |
6 Modeling Failure Probability as a Function of Load and Rate of Loading

Using logistic regression, one may estimate the probability of failure during proofloading at different load levels and rates. As seen previously, this probability is denoted by $\pi$. Once again, we start with the 2nd–order full model and subject it to backward elimination for any possible simplification. The full model is

$$\pi(\text{fail}|l, r) = \frac{1}{1 + \exp\left(-\left(\theta_0 + \theta_1 \cdot l + \theta_2 \cdot r + \theta_3 \cdot l^2 + \theta_4 \cdot r^2 + \theta_5 \cdot l \cdot r\right)\right)},$$  \tag{8}$$

where $l$ is the proofload level and $r$ is the rate. The main results for what we believe to be the best model are shown in the two tables below.

Table 13: Regression coefficient estimates, standard errors, significance-test values for the logistic model for long fingerjoints.

| Estimate | Std. Error | z value | Pr(> |z|) |
|----------|------------|---------|-------|
| (Intercept) | -21.6039 | 6.0203 | -3.59 | 0.0003 |
| load | 6.0918 | 2.2620 | 2.69 | 0.0071 |
| rate | -0.1327 | 0.0418 | -3.18 | 0.0015 |
| load$^2$ | -0.4228 | 0.2102 | -2.01 | 0.0443 |
| rate$^2$ | 0.0008 | 0.0005 | 1.69 | 0.0907 |
| load-rate | 0.0136 | 0.0052 | 2.64 | 0.0083 |

Table 14: Regression coefficient estimates, standard errors, significance-test values for the logistic model for short fingerjoints.

| Estimate | Std. Error | z value | Pr(> |z|) |
|----------|------------|---------|-------|
| (Intercept) | 111.6472 | 5604.1930 | 0.02 | 0.9841 |
| load | -55.3110 | 2595.6261 | -0.02 | 0.9830 |
| rate | 0.1299 | 0.0429 | 3.03 | 0.0024 |
| load$^2$ | 6.6539 | 294.9575 | 0.02 | 0.9820 |
| load-rate | -0.0310 | 0.0092 | -3.36 | 0.0008 |

It is natural to assume that $\pi$ increases with proofload level, and this is indeed evident from figure 11: the plots of predicted $\pi$ (via estimation equation (8)) against proofload stress levels at three different rates. One important conclusion here is that for short-fingerjoint lumber, the probability of proofload failure is estimated to be $\approx 100\%$ once the proofload level surpasses
5300 psi. This is true for all 3 proofload rates; as shown in Figure 11 (top plot), the $\hat{\pi}$ curves all converge to 100% around 5300 – 5400 psi. As for the long-fingerjoint lumber, the $\hat{\pi}$ is between 15% – 25% in aforementioned range of proofload stress levels. Furthermore, the long-fingerjoint failure probability decrease noticeably under higher proofload rate, i.e., when it takes longer to reach a designed proofload level.

Figure 11: Predicted proofload failure probability (y-axis) vs. proofload stress level for short fingerjoints (top plot) and long fingerjoints (bottom plot).
7 Optimizing Tension Proofloading

We now have developed regression models for estimating every component in the equation (3).

To model the log-normal parameter \( \mu \) as a function of load \( (l) \) and rate \( (r) \), we have for both fingerjoints:

\[
\hat{\mu}_{\text{long}} = 2.3506 - (2.42 \cdot 10^{-1}) \cdot l + (5.158 \cdot 10^{-3}) \cdot r + (2.729 \cdot 10^{-2}) \cdot l^2
- 0.0001 \cdot r^2 - (4.05 \cdot 10^{-4}) \cdot l \cdot r
\]

\[
\hat{\mu}_{\text{short}} = 1.2063 + (8.501 \cdot 10^{-2}) \cdot l - (8.926 \cdot 10^{-3}) \cdot r + (1.416 \cdot 10^{-4}) \cdot r^2.
\]

For the parameter \( \sigma \), the estimated regression models have forms:

\[
\hat{\sigma}_{\text{long}} = \exp(-1.2796 - 0.1695 \cdot l + 0.0009 \cdot r)
\]

\[
\hat{\sigma}_{\text{short}} = \exp(-1.2288 - 0.2834 \cdot l - 0.0345 \cdot r + 0.00855 \cdot l \cdot r).
\]

Since we fitted the regressions with \( \log(\sigma) \) as the response, we would exponentiate the linear function in order to estimate \( \sigma \). For the proofload failure probability \( \pi \), we have logistic regressions of forms:

\[
\hat{\pi}_{\text{long}} = \frac{1}{1 + \exp\left(-(-21.604 + 6.092 \cdot l - 0.133 \cdot r - 0.423 \cdot l^2 + 0.0008 \cdot r^2 + 0.0136 \cdot l \cdot r)\right)}
\]

\[
\hat{\pi}_{\text{short}} = \frac{1}{1 + \exp\left(-(-111.647 - 55.311 \cdot l + 0.1299 \cdot r + 6.6539 \cdot l^2 - 0.0310 \cdot l \cdot r)\right)}.
\]

Predicted probabilities \( \hat{\pi} \) are then plugged into equation 4 to predict the proportion of short fingerjoints in the survivor population. The equation is shown again below for ease of referencing.

\[
p(\delta, l, r) = \frac{\delta \cdot (1 - \pi_s(l, r))}{\delta \cdot (1 - \pi_s(l, r)) + (1 - \delta) \cdot (1 - \pi_l(l, r))}.
\]

As mentioned above, the original short-fingerjoint proportion \( \delta \) can be estimated through prior knowledge, such as historical data on the mill that produced the lumber batch in question. In our case, it is \( \delta = 0.2 \) as designed in the experiment.

In the end, we would assemble predictions from the aforementioned equations into

\[
f_X(x|\delta, l, r) = \hat{p}(\delta, l, r) \cdot f_{\text{short}}(x|\hat{\mu}_{\text{short}}(l, r), \hat{\sigma}_{\text{short}}(l, r)) +
(1 - \hat{p}(\delta, l, r)) \cdot f_{\text{long}}(x|\hat{\mu}_{\text{long}}(l, r), \hat{\sigma}_{\text{long}}(l, r)).
\]

Above function can provide us with the theoretical tensile stress distribution for survivors of any combinations of \( l, r \) and \( \delta \): proofload stress level and loading rate, and initial (pre-proofload) proportion of short-fingerjoint/defect. As such, we refer to (9) as the optimization
model of tension proof loading. Once again, one should be careful not to extrapolate the optimization model at \( l \) and \( r \) outside of our experimental range.

To assess how well our proof load optimizer models the distribution of survivors, we estimated their distributions at the experimental load and rate values, i.e., combinations of \( l = (3.8, 5.0, 6.1) \) psi (remember that all our modelling is done in 1/1000th scale) and \( r = (0.2, 6, 60) \) sec. Then the distribution is plotted over respective data histogram. As one can see in figure 12, for all 9 proofload conditions, our optimization model (9) does an admirable job of delivering distributions that fit nicely over the data histograms. In other words, our estimated distributions correspond well with the sample distribution of actual data. The aim of developing the proofload optimization model is to give researchers an idea on what the survivor distribution would look like under uncertain/untried proofload condition.

![Figure 12: Comparison of distributions estimated by 9 with data histogram at 9 proofload conditions.](image)

\( x - axis \) is tensile stress in 1000 psi and \( y - axis \) is the density.

In Section 4, we compared the tensile stress distribution of the control lumber with the stress distribution of survivors from the 9 proofload settings of our experiment. We conclude from the results that damages sustained by survivors can be offset by the deletion of weakest lumber from the original batch; meaning a “rightward” shift of the distribution’s lower tail offsets a “leftward” shift elsewhere in the distribution, making the survivors a stronger lumber population than the controls. There are various ways to construct a numerical measure that
compares the expected tensile stress of survivors and controls. An intuitive statistics measure may be the one that quantifies the probability of event: a randomly chosen lumber from a survivor population is stronger than a random lumber from the control population.

Using the optimization model, we may estimate such a probability for combinations of $l$, $r$ and $\delta$. Denote the random variable $X$ as the tensile stress of the survivors and $Y$ as that of the controls (non-proofloaded lumber), the aforementioned probability of a random survivor stinger than control is

$$p(X \geq Y) = 1 - \int_{Y} F_X(y|Y=y) \cdot f_Y(y) dy.$$ 

It is integrated over the range for $Y$, which is $\{1.948, \infty\}$ under the 3-parameter Weibull (from section 3) and $(0, \infty)$ under log-normal modeling for $f_Y(y)$. In theory, the greater this probability is, the stronger the survivor. If the probability is above 0.5, one may conclude that the survivors are stronger than controls in general. If the opposite $p(X \geq Y) < 0.5$ is true, then one may conclude that survivor is generally weaker, which implies that the improvements in the lower tail fails to offset damages sustained by the survivors. Figure 14 shows $p(X \geq Y)$ under five different rates over a range of load levels, and figure 13 shows a similar plot over a range of rate values at the 3 load levels of our experiment. Both plots shown use a Weibull-mixture model to model the controls. In addition, the initial proportion of short-fingerjoints is set at $\delta = 0.2$ or 20%, the same proportion designed for our experiment.

Based on our optimization model (9), the probability $p(X \geq Y)$ is greatest at $r = 25 \sim 30$ seconds for the higher load levels, where the probability curves in figure 13 reached maximum. Such results are also reflected in figure 14, where at rate $= 25$sec, the probability is greater throughout the load levels ranged between $3800 \sim 6100$psi. However, a problem is that the proofload rate of 25s is not financially viable in a real production setting (too time consuming). Prior works from FPInnovations found that the rate of 1 second is a satisfactory compromise between quality and economy, hence in figure 14, we also plotted the estimated probability $p(X \geq Y)$ over $l$ at rate $= 1$sec. As shown, the rate of 1sec is not too different from $r = 0.2$sec, but an improvement over $r = 6$sec at higher proofload stress levels. We would like to emphasize that these conclusions are drawn based our optimization model and using $p(X \geq Y)$ as the metric of lumber quality. It is crucial to bear in mind of this fact when applying and interpreting the analyses presented in this section.

### 7.1 $p(X \geq Y)$ at Various Initial Short Fingerjoint Proportions

An useful and informative follow-up study is to analyze the benefit of tension proofloading (using $p(X \geq Y)$ as the standard) under various $\delta$, the initial proportion of short fingerjoints in the lumber population. This analysis is implemented by estimating the $p(X \geq Y)$ under
Figure 13: Estimated probability $p(X \geq Y)$ vs. rate. $X$ denotes survivor tensile stress and $Y$ denotes the control. The control population is modelled by a 3-p Weibull.

few distinct scenarios:

1. $\delta = 0$: no short fingerjoints in the lumber population. In a real-world production setting, this means that the produced lumber are devoid of unusually weak or defective pieces.

2. $\delta = 0.1$, 0.2 and 0.3: the lumber population have moderate to noticeable proportion of weaker and/or defective sub-population.

3. $\delta = 0.4$ and 0.5: the population of fingerjointed lumber have significant proportion of weak and/or defective lumber pieces.

Figures 15, 16 and 17 show the plots of $p(X \geq Y)$ vs. $\delta$ under the proofloading load and rate settings $l = \{3800, 5000, 6100\}\ psi$ and $r = \{0.2, 1, 6, 25, 60\}\ seconds$. The initial short-joint proportion $\delta$ varies between 0 to 0.5. These figures show that, using $p(X \geq Y)$ as a benchmark, the benefit of proofloading (stronger survivors) increases with increasing $\delta$. This is an expected outcome in lieu of our earlier conclusions that tension proofloading removes weaker pieces within a lumber population (removing the lower tail): the proofload becomes more beneficial the larger the number of weaker-lumber pieces are there to remove.
In addition to above mentioned general result, we also identified few important details. First, across the range of $\delta$, the probability $p(X \geq Y)$ is higher under a higher proofloading stress. At $l = 6100\,psi$ and with the exception of $r = 60\,sec$, the survivors are expected to be stronger than the controls ($p(X \geq Y) > 0.5$) regardless of initial short-fingerjoint proportion (Figure 17). This result leads to an interesting conclusion that, at a high-enough proofload stress level, the tension proofloading is beneficial even without the presence of short-fingerjoints (or defects). To explain this idea from the perspective of lumber-stress distribution $f_X(x)$: a high-stress proofloading is capable of destroying enough lumber on the lower tail of $f_X(x)$ to offset any leftward shift of the distribution (result of weakened survivors), making proofloading potentially beneficial even for a homogenous lumber group without defects.

At the lower stress levels $l = 5000\,psi$ and $3800\,psi$, the benefit of tension proofloading is not noticeable until $\delta > 0.3$, implying that the aforementioned offset between “lower tail removal” and “survivor weakening” is difficult to achieve at a relatively low-stress proofloading. In addition, at high stress levels $l = 5000$ and $6100\,psi$, the survivors are weakest at $r = 60\,sec$ and strongest at $r = 25\,sec$. This once again illustrates the non-linear relationship between $p(X \geq Y)$ and proofloading rate, as well as the duration-of-load effect.

Figure 14 shows for the case where $\delta = 20\%$ (the experimental setting), the estimated variation pattern of $p(X \geq Y)$ over a range of stress load at rates $r = \{0.2, 1, 6, 25, 60\}\,seconds$. 

---

**Figure 14:** Estimated probability $p(X \geq Y)$ vs. load. $X$ denotes survivor tensile stress and $Y$ denotes the control. The control population is modelled by a 3-p Weibull.
Figure 15: The Probability of survivors stronger than the controls \( p(X \geq Y) \) plotted over \( \delta = 0.0 \) to 0.5, at proofloading stress \( l = 3800 \text{ psi} \) and the rates \( r = \{0.2, 1, 6, 25, 60\} \text{ seconds} \).

Following Figures 18, 19 and 20 show the same type of plot for \( \delta = 0\%, 10\%, 30\%, 40\% \) and 50\%.

These plots provide additional insights on the effects proofloading, which can be summarized as below:

1. With the exception of \( \delta = 0\% \), the survivor strength measure \( p(X \geq Y) \) exhibit monotonically increasing behaviour against proofload stress level, i.e., the proofload survivors are statistically expected to be stronger at higher load levels.

2. At proofload rate \( r = 25 \text{ seconds} \), the proofload survivors are expected to be stronger than the controls \( (p(X \geq Y) > 0.5) \) at any combination of tension proofload parameters between \( \delta = 0\% \sim 50\% \) and load levels \( l = 3800 \sim 6100 \text{ psi} \).

3. At approximately \( \delta \geq 30\% \), the tension proofloading is expected to be beneficial, i.e, creating a stronger lumber population than the lumber without proofloading. This is illustrated by Figures 19 and 20, where the curves of \( p(X \geq Y) \) vs. \( l \) lie at or above 0.5 across a range of proofloading rate.
Figure 16: The Probability of *survivors stronger than the controls* $p(X \geq Y)$ plotted over $\delta = 0.0$ to $0.5$, at proofloading stress $l = 5000\text{psi}$ and the rates $r = \{0.2, 1, 6, 25, 60\}$ seconds.

Figure 17: The Probability of *survivors stronger than the controls* $p(X \geq Y)$ plotted over $\delta = 0.0$ to $0.5$, at proofloading stress $l = 6100\text{psi}$ and the rates $r = \{0.2, 1, 6, 25, 60\}$ seconds.
Figure 18: Estimated probability $p(X \geq Y)$ vs. load at $\delta = 0\%$ (top) and $\delta = 10\%$ (bottom). $X$ denotes survivor tensile stress and $Y$ denotes the control. The control population is modelled by a 3-p Weibull.
Figure 19: Estimated probability $p(X \geq Y)$ vs. load at $\delta = 30\%$ (top) and $\delta = 40\%$ (bottom). $X$ denotes survivor tensile stress and $Y$ denotes the control. The control population is modelled by a 3-p Weibull.
Figure 20: Estimated probability $p(X \geq Y)$ vs. load at $\delta = 50\%$. $X$ denotes survivor tensile stress and $Y$ denotes the control. The control population is modelled by a 3-p Weibull.
8 Tension Proofload of Multi-Fingerjoint Lumber

Previous sections analyzed the tension proofload of single-fingerjoint lumber. In real-world market place, fingerjointed lumber will have more than one fingerjoint. A piece of multi-fingerjoint lumber is produced by gluing together multiple “blocks” of lumber pieces, where individual blocks vary between 1 to 4 feet in length with finger slots cut into their ends. Figure 21 illustrates what a piece of multi-fingerjoint lumber looks like. During production, 50ft ribbons are initially produced and subjected to tension proofloading. The ribbons are then cut into 10ft multi-jointed lumber. The number of finger joints and their locations are random.

Figure 21: An example of multi-joint lumber, where the locations and number of joints are random.

All experiments in this study are based on single-jointed lumber, so no explicit lumber destruction and tensile stress data are available for multi-jointed lumber. In order to analyze the effect of tension proofloading on multi-fingerjoint lumber, we will draw statistical inference using tensile stress distributions of single-jointed lumber fitted in Section 2. Hence, we propose to statistically predict the effect of proofloading on multi-jointed lumber based on what we know from single-joint proofloading.

8.1 Proposed Statistical Analysis

A fingerjoint can be a short or long fingerjoint, and in this experiment, a short joint represents a “defect” in the ribbon.

In Section 2, we estimated the tensile stress distributions of single-joint short and long-fingerjoint lumber from both the control group and proofload survivors. Here, let $X$ denote a random variable representing lumber tensile stress (unit psi) with superscript $s$ and $l$ denoting the fingerjoint length, and subscript $c$ and $s$ denoting the controls and survivors. So $X^s_c$ denote the tensile stress of short-fingerjoint control.

In this section, we will present two general method for analyzing the effect of proofloading on multi-jointed lumber, both will be based on the already fitted tensile stress distributions of single-jointed lumber. One analysis is based on tensile stress comparison between control and survivors, the other method analyzes the survival ratio and probability of lumber under various proofload conditions.
8.2 Analysis I: Comparison of Multi-joint Lumber Tensile Stress

Here, the following question is examined: Suppose there are two batches of 10ft multi-jointed lumber each with \( n \) joints, one batch did not receive proofload (the control) and one batch did (survivors). What is the probability that a randomly chosen piece of lumber from each group is weaker than a pre-determined stress load? In other words, which lumber group has higher design value?

Assume the tensile stress of each joint is independent of each other, we can derive a joint distribution representing the tensile stress of a \( n \)-jointed lumber with \( n_{\text{short}} \):

\[
X^m_c = \frac{n_{\text{short}}}{n} \prod_{i=1}^{n_{\text{short}}} (X^s_{c,i}) + \frac{n - n_{\text{short}}}{n} \prod_{j=1}^{n-n_{\text{short}}} (X^l_{c,j}),
\]

\[
X^m_s = \frac{n_{\text{short}}}{n} \prod_{i=1}^{n_{\text{short}}} (X^s_{s,i}) + \frac{n - n_{\text{short}}}{n} \prod_{j=1}^{n-n_{\text{short}}} (X^l_{s,j}).
\]

(10)

where superscript \( m = \{s, l\} \) (short or long fingerjoint).

Since failure at one joint is equivalent to the failure of entire ribbon, the lumber failure probability can be calculated as the probability of at least one joint failing below a stress load:

\[
P(\text{at least one } X^m_c \text{ failure}) = 1 - \left[ \frac{n_{\text{short}}}{n} P(X^s_c \geq \text{load})^{n_{\text{short}}} - \frac{n - n_{\text{short}}}{n} P(X^l_c \geq \text{load})^{n-n_{\text{short}}} \right],
\]

\[
P(\text{at least one } X^m_s \text{ failure}) = 1 - \left[ \frac{n_{\text{short}}}{n} P(X^s_s \geq \text{load})^{n_{\text{short}}} - \frac{n - n_{\text{short}}}{n} P(X^l_s \geq \text{load})^{n-n_{\text{short}}} \right].
\]

(11)

For survivors, if the proofload process did its job and destroyed all defects, the probability simplifies to: \( P(\text{at least one } X^m_s \text{ failure}) = 1 - P(X^l_s \geq \text{load})^n \). Results can be obtained at combinations of \( n \) and \( n_{\text{short}} \) over a range of load, and plotted to compare the tensile stress properties of different lumber groups.

It was shown in Section 2 that \( X_s \)'s have stronger profiles than \( X_c \)'s. The description “stronger” corresponds to higher design values and the fact that a random lumber piece from the survivor group tends fail at higher stress load than a random piece of control lumber. Therefore, we can argue logically using (11) that \( P(\text{at least one } X^m_s \text{ failure}) \geq P(\text{at least one } X^m_c \text{ failure}) \) at any stress load.

Alternatively, one may define the tensile stress of a multiple-joint lumber using its “weakest link”. Given \( n \), \( n_{\text{short}} \) and individual \( X \)'s, we can derive using theories of ordered statistics, tensile stress at the weakest joint \( X^{(1)} \) for both the controls and survivors. These stress distributions can then be compared via statistical measure \( P(X^{(1)}_{\text{control}} \leq \text{load}) \) and \( P(X^{(1)}_{\text{survivor}} \leq \text{load}) \).
However, this alternative lumber stress measure is not used in following analyses.

### 8.2.1 Results of Analysis I

Current Canadian standards specify 1.33/2.1x tensile strength, which translates to proofload stress loads between 600psi and 1400psi depending on lumber size tested. The US standards specify 1.5/2.1x tensile strength, but only for certain products such as I-joist flanges. To make existing short fingerjoint data better emulate the stress profiles of defective/weak jointed lumber, we scale short fingerjoint distributions to 1/4. As an example, Figure 22 shows the data histogram of short-jointed control scaled to 1/4.

![Short-fingerjoint control lumber scaled 1/4](image)

**Figure 22: Data histogram of short-fingerjointed control lumber scaled to 1/4.**

Figure 23 shows the failure probabilities $P(\text{at least one } X^m_s \text{ failure})$ (control) and $P(\text{at least one } X^m_l \text{ failure})$ (surviver) between stress loads (500, 2000)psi. The control probabilities are plotted for $n = 3$ joints with $n_{short} = 1$ and $n_{short} = 3$, and the survivor probabilities are plotted for $n = 3$ joints with no defects. The individual distributions of $X_s$’s and $X_l$’s are the Log-Normal distributions fitted in Section 2, and the A1 survivor distribution is used.

Since the purpose of tension proofloading is to identify/destroy defective lumber from production, $n_{short} = 0$ should be the ideal representation of proofloaded lumber. As shown, at $n_{short} = 0$, survivors have near 0 probability of failing below the stress loads in the range 500 $\sim$ 2000psi. This is also true when the control contains no weak or defective joints (not shown).

However, tension proofloading becomes an effective quality control tool when $n_{short} > 0$, and its usefulness increases dramatically when $n_{short}$ increases. When $n_{short} = 1$ (top Figure
The control lumber have 40% failure probability at stress load of 1250 psi or more. When the number of weak/defective joints increases to 3 (bottom Figure 23), the failure probability is almost 100% when \( \approx 1200 \) psi of load is applied.

Figure 24 shows the probabilities of \( P(\text{at least one } X^m_c \text{ failure}) \) and \( P(\text{at least one } X^m_s \text{ failure}) \) the stress loads of 500 psi and 2000 psi. The control probabilities are calculated at \( n = n_{\text{short}} = 3 \) and the surviver probabilities are calculated at \( n = 3 \) with \( n_{\text{short}} = 0, 1, 2, 3 \). The cases \( n_{\text{short}} = 1, 2, 3 \) represent the situations where proofloading failed to do its job: at least 1 out of 41
of the 3 possible defects survived proofloading. As shown, the failure probability of proofload survivors increases noticeably when weak or defective joints survived proofloading. When comparing the tensile stress profiles of control vs. survivors at \( n_{\text{short}} = 3 \), we see that the survivors are still stronger than corresponding control around the range \( \text{load} = 1000\psi \). This is because any short-fingerjoint survivor represents the “stronger of defective/weak joints”.

Figure 24: Failure probabilities at stress loads between 500 and 2000 \( \psi \). Control probabilities \( P(\text{at least one } X^m_c \text{ failure})'s \) are calculated for \( n = n_{\text{short}} = 3 \), and survivor probabilities \( P(\text{at least one } X^m_s \text{ failure})'s \) are calculated at \( n = 3 \) and \( n_{\text{short}} = 0, 1, 2, 3 \).

8.3 Analysis II: Survival Ratio and Probability of n-joint Ribbon

A systematic way of describing the tensile stress of n-joint ribbon can be based on the survival ratio and survival probability:

\[
\begin{align*}
  r_n(l) &= \frac{S_n(l)}{S_0(l)} = \exp\{-n \times [1 - S_1(l)]\} \quad \text{and,} \\
  S_n(l) &= r_n(l) \times S_0(l).
\end{align*}
\]

Here are useful details regarding (12):

- \( S_0(l), S_1(l) \) and \( S_n(l) \) are respectively, the survival probabilities of no-joint, single-joint and n-joint ribbon at stress load \( l \). The probability is defined as \( P(X \geq l) \), the lumber piece’s tensile stress is \( \geq l \), thereby withstanding the applied stress load.
• This ratio describes the rate in which $S_n(l)$ decrease from $S_0(l)$, or how much weaker a
n-joint lumber is expected to be when compared against a solid piece of lumber.
• This ratio decreases exponentially as $n$ increases. When $n = 0$, the ratio is 1.
• This ratio increases with increasing $S_1(n)$. So higher the tensile stress at a single-joint,
the closer the stress profile of n-joint ribbon is to a no-joint lumber.

In the presence of defective or weak fingerjoints, there are also $n_s$: the number of short
fingerjoints.

8.3.1 Results of Analysis II

To visualize the tensile stress profile of n-joint lumber, one can plot how $S_n(l)$ varies with $l$ and
$n$ (Figure 25). Here, the long-fingerjoint data can be used as proxy for normal or high-quality
single joints, then visualize how $S_n(l)$ vs. $l$ profile change with $n$. Since no short-joints are
used in calculation, the destructive stress loads are higher than the earlier analysis. As shown,
the survival probability start to go below 1 when the applied stress load increases beyond
$\approx 4000\text{psi}$ for A1, $\approx 4500\text{psi}$ for B1 and $\approx 5000\text{psi}$ for C1. These are the points one may
start to expect lumber failure. Moreover, the survival probability decreases noticeably as $n$
increases.

Figure 26 visualizes for multi-joint A1 survivors, the survival probability and ratio at
$n = 5$ joints with increasing number/proportion of short fingerjoints. We see that when short-
fingerjoints survives proofloading, the survival probability/ratio start to drops off noticeably
near a relative low stress load of 1000psi. These results reflect prior analysis in this section:
when supposedly defective joints slip through proofloading, the survivors are expected to
display marginal improvement over non-proofloaded controls. In the end, one may view the
survival function (12) as a way of systematically describing the analyses in this section.
Figure 25: Survival probabilities of A1, B1 and C1 lumber at stress loads between 3000 and 6100 psi given various number of joints.
Figure 26: Survival ratio and probability of A1 lumber at $n = 5$ joints and increasing number of short (defective) joints.
9 Conclusions

This report contains 3 main statistical analysis of tension proofloading on fingerjointed lumber. The following are their summary and conclusions.

9.1 The benefit and trad-off of tension proofloading

The primary objectives of this report are to determine first, whether tension proofloading is beneficial and second, if proofloading is indeed a viable quality control method, how can one optimize the procedure. The first question was born out of the concern that, based on empirical analysis on the sample data, certain proofload survivors sustained damages. Therefore one must determine whether the discarded defects/short-fingerjoints are enough to offset the damages and make proofload survivors stronger as a population.

This question was answered by estimating the tensile stress distributions of control (non-proofloaded) and survivor (proofloaded) populations. Here, the distributions were estimated separately for short and long fingerjoints and combined into a mixture distribution. By comparing these stress distributions, we concluded that despite the existence of damages, the stress distributions of survivors have higher low-quantile values and heavier density in the high-stress region. In other words, if one were to pick a random piece of lumber from a survivor population, the probability of this piece of lumber having high tensile stress is higher than doing so for the control population. The number of short-fingerjoints discarded increases with the proofload stress level, and as the load reaches $l = 6000$psi, the survivor would only contain long-fingerjointed lumber, albeit with the risk of damaging survivors. Furthermore, we noticed that given the damages and unavoidable experimental errors, the survivor distributions are best represented by a regular distribution instead of a conditional distribution.

9.2 Statistical optimizer of proofloading

Having established the merit of proofloading, the logical next step would be to optimize this process. Such analysis provides additional insights on the expected outcome and effectiveness of tension proofloading. Namely, how our self-defined proofload benefit measure $p(\text{survivors} > \text{controls})$ varies over a range of interactions between the proofloading load and rate ($l$ and $r$), and initial short-fingerjoint (or defective lumber) proportion $\delta$.

The proofloading optimization analysis is implemented by building a regression model that, given a set of load and rate values, predicts the distribution parameters and the probability of failure during proofloading. Given an initial proportion of short-fingerjoint (or defective) lumber, these predictions can be constructed into a mixture distribution describing a theoretical tensile stress variation of survivors from any proofload specification. The estimation model we developed corresponded well with our exiting lumber data (described in Section 2). The
results have shown that using \( p(\text{survivors} > \text{controls}) \) as a benchmark, tension proofloading is optimized, i.e., statistically expected to create the strongest survivors, at load \( \geq 5300 \text{psi} \) and rate \( \approx 25 \text{seconds} \). The probability \( p(\text{survivors} > \text{controls}) \) mostly exhibits a monotonically increasing response to the variation in \( l \), while its relationship to varying \( r \) is non-linear.

Above mentioned proofload-optimization analysis also provided additional insights on the benefit and effectiveness of lumber tension-proofloading. One result indicate that a high-stress proofload, e.g., \( l = 6100 \text{psi} \), the survivors are expected to form a stronger lumber population than the controls even with the absence of any short-fingerjoints or defects. This result reiterated our earlier finding that tension proofloading can be fine-tuned to remove enough weaker lumber pieces in a population to offset any potential damages inflicted onto the survivors.

In the end, one should note that in our study, the short-fingerjoint lumber are used to represent the weak sub-population which in reality, is represented by defective lumber. In real-world marketplace, however, the type of short-fingerjoints we used is considered high-quality enough to be legitimate lumber product. This explains the reason behind the relatively high proofload stress levels needed for statistical analyses. In general, a proofload stress of \( \leq 3000 \text{psi} \) is sufficient to detect and remove defects from circulation (personal communications with Ciprian, 2014), ?, ?). Regardless of the type of fingerjointed lumber used to represent weak lumber pieces, the central message drawn from our analyses is that: with proper optimization, tension proofloading is a viable production technique for removing weak and/or defective finger-jointed lumber and creating a stronger (higher-quality) lumber population for sale.

### 9.3 Tension Proofloading of Multi-fingerjoint Lumber

The last analysis uses the fitted tensile stress distributions of single-joint lumber to make inference on the effectiveness of proofloading on multi-joint lumber. Results showed that under the presence of one or more weak/defective fingerjoints, a multi-joint lumber without the proofload treatment would fail with varying probability, under relatively light stress loads between \( 500 \sim 2000 \text{psi} \). The failure probability starts from a not insignificant value and quickly increases to 1 when the number of weak/defective joints increases. The above mentioned stress loads cover existing range of proofload standards, hence failure at these stress loads would be considered a product failure.

A piece of proofloaded lumber is expected to fail at (or survive till) a much higher stress load. Depending on the proofload settings, the failure stress load starts at approximately \( 4000 \text{psi} \) to \( 5000 \text{psi} \). However, the effectiveness of proofloading is significant only when the procedure is done properly, i.e., all weak/defective joints are destroyed. Otherwise, using survival/failure probability as measures, the proofloaded lumber are expected to have very marginal quality improvement over non-proofloaded ones.

There are certainly other approaches to answer above questions. The methods proposed
here are what we determined as the simplest and most direct analyses using available data and results from previous sections.
Appendix 1: R Code for Estimating a 3-Parameter Weibull

We first created a function called `fit.weibull` that creates a 3-p Weibull log-likelihood given a data (data.s in code), then minimizes the negative of log-likelihood (maximizing log-likelihood). Sentences following # are in-code comments.

```r
fit.weibull=function(data.s,p.start)
{
  #Log-likelihood of 3-parameter Weibull.
  ll=function(p,data)
  {
    x1=(data-p[3])/p[2]
    f=(p[1]/p[2])*(x1^(p[1]-1))*exp(-x1^p[1])
    -sum(log(f))
  }
  p=p.start #Starting values for minimizing the negative log-likelihood.
  #Function nlm minimizes negative log-likelihood.
  out=nlm(ll,p,hessian=T,data=data.s)
}

Using `fit.weibull`, we created another function `bf` output that delivers the estimations $\hat{\alpha}$, $\hat{\beta}$ and $\hat{c}$, their standard errors and the maximized log-likelihood values.

```r
output=function(data,name,p.start)
{
  fit=fit.weibull(data,p.start)
  est=fit$estimate
  #Invert Hessian matrix, and the squared-root of its diagonal elements are the
  #standard errors of the parameter estimates.
  acov=solve(fit$hessian)
  se.1=sqrt(acov[1,1])
  se.2=sqrt(acov[2,2])
  se.3=sqrt(acov[3,3])
  loglik=-fit$minimum
  cat("loglik, estimation and se for",name,"\n")
  #The output consists of: 3 parameter estimates, their standard errors,
  #and the maximized log-likelihood value
  out=c(est[1],est[2],est[3],se.1,se.2,se.3,loglik)
```

Appendix 2: Modeling the Survivors using a Conditional Distribution

Note: The analyses presented in Appendix 2, 3 and 4 are summarized from our earlier studies. Instead of scaling the tensile stress data by 1/1000th as in the main report, these results are based on data scaled to 1/100th, which gives estimated parameter at a different scale. This however, does not influence the conclusions drawn.

It is natural to consider that survivor populations formed by the strongest lumber from a pre-proofload batch, if \( f_X(x) \) denotes the strength distribution of this pre-proofloaded lumber, then the survivors may be described by \( f_X(x|x > l) = f_X(x)/F_X(l), \ x > l, \) where \( l \) is the proofload stress level. The shape of the distribution is illustrated in figure 27; the original mixture distribution (blue curve) has a lower tail, whereas the survivor distribution (green curve) is a truncated (at \( l = 3800 \) in the plot) and normalized distribution. The tensile strength data on the treatment groups contain two types of stress values, for those failed proofloading, it is the stress value at the time of proofload failure, while for the survivors, the stress values are those recorded during “test-to-failure” procedure following the proofload.

![Figure 27: Illustration of a truncated distribution describing the survivors](image)

Unfortunately, estimating pre-proofload strength distribution \( f_X(x) \) won’t be achieved by simply combining the two stress data and fitting certain types of pdf. In reality, the experiment has shown that certain survivors had tensile strength less than the proofload level \( l \), showing the possibility of damage through proofloading. In fact, for all 3 loading rates of level B, the maximum strength of the failures are greater than the minimum strength of the survivors. It is not reasonable to assume two types of stress data (failures and survivors) come from the same
lumber population. However, if it were possible to proofload the same batch of lumber twice, we would expect to see the same set of failures. Now consider the Weibull based log-likelihood

\[ L \propto \log(F(l)^{n_f} \prod_{i=1}^{n_p} \left( \frac{b}{a} \left( \frac{x_i - c}{a} \right)^{b-1} \exp \left( - \left( \frac{x_i - c}{a} \right)^b \right) \right)), \tag{13} \]

where \( n_f \) and \( n_p \) are the numbers of failures and survivors in the sample, \( x_i \)'s are the tensile strength of the survivors and \( l \) is the proofload stress level. If we let \( l \geq c \) be another parameter to be estimated, then the MLE \( \{\hat{a}, \hat{b}, \hat{c}\} \) from the above likelihood would gave a Weibull distribution \( f_X(x) \) that describes a theoretical batch of lumber weakened by proofloading. The MLE \( \hat{l} \) may be viewed as the theoretical load level that would “cut” the weakened lumber batch into \( n_f \) failures and \( n_p \) survivors. The strength distribution of the survivors can be subsequently derived as

\[ f_X(x| x > \hat{l}) = \frac{f_X(x)}{P_X(x > \hat{l})} = \left( \frac{1}{F_X(\hat{l})} \right) f_X(x), \quad x > \hat{l}. \]

Performing the aforementioned procedures separately for short and long fingerjoints, one can arrive at the mixture distribution for the survivors

\[ f_X(x) = \left( \frac{1}{F_{\text{short}}(\hat{l})} \right) f_{\text{short}}(x) \cdot p + \left( \frac{1}{F_{\text{long}}(\hat{l})} \right) f_{\text{long}}(x) \cdot (1 - p), \]

where \( p \) is from the survivor samples and the distribution has support \( x > \hat{l} = \min(\hat{l}_{\text{short}}, \hat{l}_{\text{short}}) \).

This method models the survivor distributions quite well given that \( n_f \) is small. However, this method proved deficient in modeling the data with a high number of failures. Short fingerjoints in level B and all of level C has a failure rate exceeding 30%, reaching 80% in case of B1 and B2. Since the result of this method is a conditional distribution derived from a Weibull pdf, excessive \( n_f \) means that \( f(x|x > \hat{l}) \) would only describe the “upper-tail” of a distribution and this is evident from the figure 28 and 29. The implication of such “tail” distribution is that over the support of \( f(x|x > \hat{l}) \) (the survivor data), our resultant model would over-estimate the density on the “lower strength region” of the dataset and make the survivor population seem weaker than it really is.

The lack-of-fit exhibited by this conditional method is best explained by examining the data histograms, where one would observe noticeable lower-tail distribution instead of distributions with “cleanly-cut” lower end (as in 27). This implies that strength of the survivors form a new pdf instead of a conditional pdf as originally anticipated. There are a few explanations for such phenomenon. As mentioned, surviving lumber sustain damages through proofloading. Therefore the weakest amongst the survivors would have tensile strength gradually less than the proofload stress level, creating a lower-tail distribution. The matter would be complicated
further by the unavoidable measurement and machine errors; the testing machine cannot perfectly achieve the desired stress load at a designed rate, thus creating random variations across the tail of distributions.

Appendix 3: Details of Chi-squared Test

This section gives an example of the Chi-squared test procedure for the long fingerjoints of control.

The sample data is first divided into \( k = 11 \) bins with intervals as such

Given the cdf of an distribution, the expected count in each bin \( i \) is calculated as \( E_i = (\hat{F}(b_i) - \hat{F}(a_i)) \cdot N \), where \((a_i, b_i)\) is the interval for bin \( i \), \( N \) is the sample size and function \( \hat{F} \) is the estimated cdf. For example, under the test for 3-parameter Weibull,

\[
\hat{F}(x) = 1 - \exp \left( - \left( \frac{x - \hat{c}}{\hat{a}} \right)^{\hat{b}} \right).
\]
Table 15: Interval for each bins

<table>
<thead>
<tr>
<th>bin</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>interval</td>
<td>33.24, 48.70</td>
<td>48.70, 53.17</td>
<td>53.17, 57.63</td>
<td>57.63, 60.83</td>
</tr>
<tr>
<td>bin</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>interval</td>
<td>60.83, 64.00</td>
<td>64.00, 66.55</td>
<td>66.55, 69.63</td>
<td>69.63, 72.98</td>
</tr>
<tr>
<td>bin</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>interval</td>
<td>72.98, 75.91</td>
<td>75.91, 79.10</td>
<td>79.10, 90.56</td>
<td></td>
</tr>
</tbody>
</table>

The expected count $E_i$’s under the 3-parameter Weibull and observed count $O_i$’s (count of sample data in each bin) are as shown

<table>
<thead>
<tr>
<th>bin $i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_i$</td>
<td>13</td>
<td>14</td>
<td>22</td>
<td>21</td>
<td>24</td>
<td>21</td>
<td>26</td>
<td>26</td>
<td>19</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>$O_i$</td>
<td>14</td>
<td>9</td>
<td>20</td>
<td>21</td>
<td>23</td>
<td>25</td>
<td>34</td>
<td>26</td>
<td>18</td>
<td>12</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 16: $E_i$ and $O_i$

The test statistic $X^2 = \sum_{i=1}^{11} (O_i - E_i)^2 / E_i$ is 6.0247. The test statistic has approximately a Chi-squared distribution with degrees of freedom $df = k - p - 1 = 7$, where $k = 11$ and $p = 3$ is the number of parameters estimated. Our null assumption $H_0$ here is that our estimated 3-parameter Weibull corresponds well with the actual data, $p-value$ is 1 minus the cdf of $X^2 = 6.0247$ under distribution $\chi^2_7$, and it is interpreted as the probability of obtaining data at least as extreme as the one we have assuming $H_0$ is true. Therefore, a high $p-value$ implies that our fitted distribution corresponded well with the data, and it is a good representation of lumber strength distribution. We decided that at $p-value > 0.05$, we fail to reject null assumption $H_0$ and conclude sufficiently good fit. The $p-value$ is 0.53687 ≫ 0.05 in our case.

Appendix 4: Effect of Load and Rate on Distribution Parameters Using a Likelihood Approach

Before conducting a regression analysis, we examined the effect of load and rate on distribution parameter values. We studied the effect of proofload rate on survivor distribution parameters given a proofload stress level, and the effect of load level under a proofload rate.
3-parameter Weibull

Extending the idea of having distribution \( f_X(x) \) for a homogeneous lumber population, such as short and long fingerjoints for A1~3, B1~3 and C1~3, we could also have distributions for surviving lumber from the same load or rate. It would have log-likelihood under 3-parameter Weibull (eq. 1) of the form

\[
L_i = \sum_{i=1}^{n_{1i}} \log f(x|a_1, b_1, c_1) + \sum_{i=1}^{n_{2i}} \log f(x|a_2, b_2, c_2) + \sum_{i=1}^{n_{3i}} \log f(x|a_3, b_3, c_3), \quad i = A, B, C
\]

for survivors under a same load \( i \), for either short or long fingerjoint. \( \{n_1, n_2, n_3\} \) are the number of lumber for rate 1, 2 and 3 respectively under \( i \)-th stress load. And we have 3 Weibull distributions with differing \( \{a, b, c\} \) combined into one large log-likelihood with 9 parameters to estimate.

The idea is that we would first estimate 9 parameters using the combined data for survivors of A, B or C, then fix certain parameter to be the same for all 3 distributions; \( b_1 = b_2 = b_3 \) for shape, \( a_1 = a_2 = a_3 \) for scale, \( c_1 = c_2 = c_3 \) for location and estimate the parameters again, which in the reduced form would have 7 parameters. If the rate is not a significant factor in determining the value of certain parameters, then we should expect the likelihood of reduced model (7-p) under the MLE to be similar to that of the full model (9-p). The significance test is conducted using the likelihood ratio test. Table 17 shows the log-likelihoods under MLE from different parameter set of equation 14.

<table>
<thead>
<tr>
<th></th>
<th>9-parameter</th>
<th>shape(b) fixed</th>
<th>scale(a) fixed</th>
<th>location(c) fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-long</td>
<td>-2363.5902</td>
<td>-2363.6942</td>
<td>-2363.7004</td>
<td>-2933.5684</td>
</tr>
<tr>
<td>B-short</td>
<td>-146.8299</td>
<td>-148.2837</td>
<td>-148.7612</td>
<td>-148.6419</td>
</tr>
<tr>
<td>C-long</td>
<td>-1229.7951</td>
<td>-1231.4060</td>
<td>-1230.6432</td>
<td>-1233.3670</td>
</tr>
</tbody>
</table>

Table 17: Log-likelihood under MLE for differing parameter set of eq.14

Given that difference in parameters from the full and reduced likelihood is 2, then \( D = 2 \times (L_{full} - L_{reduced}) \) should follow \( \chi^2_2 \) distribution. There are 15 \( D \) values in total; 3 differences under each load/fingerjoint combination. We choose the critical value to be \( \chi^2_{2,0.05} = 5.991 \), so if \( D \) for a parameter under any load level \( i \) exceed 5.991, we would conclude that rate has significant effect on that parameter under load level \( i \). For example, by fixing the shape across the rate \{1, 2, 3\} under A-long, we have log-likelihood \(-2363.6942\) and \( D = 2 \times (-2363.5902 - (-2363.6942)) = 0.208 < 5.99 \), so the proofloading rate does not have significant effect on shape(b) for long-fingerjointed survivors from load A. Table 18 shows the D values, and as
we can see, rate has a significant effect on location parameters for long-fingerjointed survivors from all load values.

Table 18: $D$ values from table 17

<table>
<thead>
<tr>
<th></th>
<th>shape(b)</th>
<th>scale(a)</th>
<th>location(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-short</td>
<td>4.27</td>
<td>1.08</td>
<td>3.85</td>
</tr>
<tr>
<td>A-long</td>
<td>0.21</td>
<td>0.22</td>
<td>1139.96</td>
</tr>
<tr>
<td>B-short</td>
<td>2.91</td>
<td>3.86</td>
<td>3.62</td>
</tr>
<tr>
<td>B-long</td>
<td>5.46</td>
<td>2.65</td>
<td>671.27</td>
</tr>
<tr>
<td>C-long</td>
<td>3.22</td>
<td>1.70</td>
<td>7.14</td>
</tr>
</tbody>
</table>

Similarly, one may construct log-likelihood

$$L_j = \sum_{i=1}^{n_A} \log f(x|a_A, b_A, c_A) + \sum_{i=1}^{n_B} \log f(x|a_B, b_B, c_B) + \sum_{i=1}^{n_C} \log f(x|a_C, b_C, c_C), \quad j = 1, 2, 3 \quad (15)$$

for survivors under the same rate $j$, $\{n_A, n_B, n_C\}$ are the sample sizes for load $A$, $B$ and $C$ respectively under rate $j$. Table 19 show the log-likelihood values for the parameter sets of eq.15 and table 20 shows the $D$ values.

Table 19: Log-likelihood under MLE for differing parameter set of eq.15

<table>
<thead>
<tr>
<th></th>
<th>9-parameter</th>
<th>shape(b) fixed</th>
<th>scale(a) fixed</th>
<th>location(c) fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate.1-long</td>
<td>-1811.1352</td>
<td>-1813.3179</td>
<td>-1818.9468</td>
<td>-1819.5177</td>
</tr>
<tr>
<td>Rate.2-short</td>
<td>-162.8381</td>
<td>-163.9527</td>
<td>-164.0263</td>
<td>-164.5510</td>
</tr>
<tr>
<td>Rate.2-long</td>
<td>-1874.2077</td>
<td>-1883.3160</td>
<td>-1888.3773</td>
<td>-1890.0231</td>
</tr>
<tr>
<td>Rate.3-short</td>
<td>-217.6642</td>
<td>-217.7467</td>
<td>-218.4515</td>
<td>-217.7554</td>
</tr>
<tr>
<td>Rate.3-long</td>
<td>-1909.4319</td>
<td>-1920.7054</td>
<td>-1933.5833</td>
<td>-1931.9043</td>
</tr>
</tbody>
</table>

Table 20: $D$ values from table 19

<table>
<thead>
<tr>
<th></th>
<th>shape(b)</th>
<th>scale(a)</th>
<th>location(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate.1-short</td>
<td>1.32</td>
<td>0.22</td>
<td>0.01</td>
</tr>
<tr>
<td>Rate.1-long</td>
<td>4.37</td>
<td>15.62</td>
<td>16.77</td>
</tr>
<tr>
<td>Rate.2-short</td>
<td>2.23</td>
<td>2.38</td>
<td>3.43</td>
</tr>
<tr>
<td>Rate.2-long</td>
<td>18.22</td>
<td>28.34</td>
<td>31.63</td>
</tr>
<tr>
<td>Rate.3-short</td>
<td>0.17</td>
<td>1.57</td>
<td>0.18</td>
</tr>
<tr>
<td>Rate.3-long</td>
<td>22.55</td>
<td>48.30</td>
<td>44.94</td>
</tr>
</tbody>
</table>
Based on the table $D$, the effect of stress load is more significant on the parameter values. Besides long fingerjoints from proofload rate 1, stress load affects all parameters \{a, b, c\} for the long-fingerjointed survivors.

**Log-Normal**

The same analysis as that before for the log-normal distribution parameters are shown in the following tables. Based on the results, it seems that under a certain stress load, rate has little effect on log-normal parameter values. But under any given rate, the proofload stress level has a significant influence on the log-normal parameter $\mu$ for both fingerjoints and $\sigma$ for long fingerjoints. This type of analysis can be extended into more complex forms, such as fixing two or more parameter simultaneously.

<table>
<thead>
<tr>
<th></th>
<th>9-parameter</th>
<th>mean ($\mu$) fixed</th>
<th>sigma ($\sigma$) fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-short</td>
<td>-418.036</td>
<td>-421.349</td>
<td>-419.073</td>
</tr>
<tr>
<td>A-long</td>
<td>-2376.873</td>
<td>-2378.554</td>
<td>-2379.808</td>
</tr>
<tr>
<td>B-short</td>
<td>-148.552</td>
<td>-148.971</td>
<td>-151.037</td>
</tr>
<tr>
<td>C-long</td>
<td>-1228.638</td>
<td>-1242.210</td>
<td>-1228.752</td>
</tr>
</tbody>
</table>

Table 21: MLE log-likelihood when fixing parameters across the rate

<table>
<thead>
<tr>
<th></th>
<th>9-parameter</th>
<th>mean ($\mu$) fixed</th>
<th>sigma ($\sigma$) fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate.1-short</td>
<td>-183.481</td>
<td>-186.955</td>
<td>-184.899</td>
</tr>
<tr>
<td>Rate.1-long</td>
<td>-1816.161</td>
<td>-1823.352</td>
<td>-1831.945</td>
</tr>
<tr>
<td>Rate.2-short</td>
<td>-164.978</td>
<td>-170.322</td>
<td>-165.219</td>
</tr>
<tr>
<td>Rate.2-long</td>
<td>-1883.522</td>
<td>-1913.167</td>
<td>-1889.621</td>
</tr>
<tr>
<td>Rate.3-short</td>
<td>-218.129</td>
<td>-226.420</td>
<td>-219.362</td>
</tr>
<tr>
<td>Rate.3-long</td>
<td>-1911.261</td>
<td>-1914.812</td>
<td>-1926.735</td>
</tr>
</tbody>
</table>

Table 22: MLE log-likelihood when fixing parameters across the load
<table>
<thead>
<tr>
<th></th>
<th>mean (μ) fixed</th>
<th>sigma (σ) fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-short</td>
<td>6.63</td>
<td>2.07</td>
</tr>
<tr>
<td>A-long</td>
<td>3.36</td>
<td>5.87</td>
</tr>
<tr>
<td>B-short</td>
<td>0.84</td>
<td>4.97</td>
</tr>
<tr>
<td>B-long</td>
<td>0.89</td>
<td>1.63</td>
</tr>
<tr>
<td>C-long</td>
<td>27.14</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Table 23: $D$ values from table 21

<table>
<thead>
<tr>
<th></th>
<th>mean (μ) fixed</th>
<th>sigma (σ) fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate.1-short</td>
<td>6.95</td>
<td>2.84</td>
</tr>
<tr>
<td>Rate.1-long</td>
<td>14.38</td>
<td>31.57</td>
</tr>
<tr>
<td>Rate.2-short</td>
<td>10.69</td>
<td>0.48</td>
</tr>
<tr>
<td>Rate.2-long</td>
<td>59.29</td>
<td>12.20</td>
</tr>
<tr>
<td>Rate.3-short</td>
<td>16.58</td>
<td>2.47</td>
</tr>
<tr>
<td>Rate.3-long</td>
<td>7.10</td>
<td>30.95</td>
</tr>
</tbody>
</table>

Table 24: $D$ values from table 22