Breaking the Same Board Twice:
the Magic of Statistics!

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Outline

1 Mechanical Strength Properties of Lumber
2 Proof Load Designs
3 Penalized Single Proof Load Design
4 Conclusion
1 Mechanical Strength Properties of Lumber

Floating staircase
Earth Science Building
University of British Columbia
Mechanical Strength Properties of Lumber

- Major Mechanical Strength Properties:
  - Modulus of Rupture (MOR)
  - Ultimate Tensile Strength (UTS)
  - Ultimate Compressive Strength (UCS)

Stresses affect all strength properties simultaneously.
2 Proof Load Designs

Single Proof Load Design (Evans et. al., 1984)

Proof loading technique: testing a specimen up to a set load and passing specimens through that do not break at this load.

- Step 1: each specimen is proof loaded in X mode to a predetermined load $L_1$.
  - Break: X is observed.
  - Survive: go to Step 2.

- Step 2: survivors are tested to failure in Y mode.
\[ Z = \begin{cases} \ X \quad \text{if } X < L_1 \\ \ Y \quad \text{if } X > L_1 \end{cases} \quad \text{and} \quad U = \begin{cases} 1 \quad \text{if } X < L_1 \\ 0 \quad \text{if } X > L_1 \end{cases} \]

Maximum likelihood estimator:

- Asymptotic consistent and normal distributed.
- Poor finite sample performance.
Why poor finite sample performance?
Symmetric Proof Load Design (Amorim & Johnson, 1986)

Specimens

Group 1
Proof load in X mode,
Break in Y mode.

Group 2
Proof load in Y mode,
Break in X mode.

• Required techniques are not always possible.

• Reality inefficiency.
Model Specification:

- Consider two log-transformed destructive strength, $X$ and $Y$:

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix}
\sim N_2
\left(
\begin{bmatrix}
\mu_x \\
\mu_y
\end{bmatrix},
\begin{bmatrix}
\sigma^2_x & \rho\sigma_x\sigma_y \\
\rho\sigma_x\sigma_y & \sigma^2_y
\end{bmatrix}
\right).
\]

- Dependence: $\rho$. 
3 Penalized Single Proof Load Design

Motivation of our Penalty Function

<table>
<thead>
<tr>
<th>Likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Single proof load design)</td>
</tr>
<tr>
<td>- Proof load in X mode.</td>
</tr>
<tr>
<td>- Break in Y mode.</td>
</tr>
<tr>
<td>⇓</td>
</tr>
<tr>
<td>Observations ((x_i, y_i)'s.)</td>
</tr>
<tr>
<td>⇓</td>
</tr>
<tr>
<td>(L(\theta; x, y))</td>
</tr>
</tbody>
</table>
Motivation of our Penalty Function

Suppose hypothetically, we had conducted a single proof load design but proof loaded in Y mode to a pre-determined level \( L_2 \).

We observe hypothetically

\[
W^h = \begin{cases}
Y^h & \text{if } Y^h < L_2 \\
X^h & \text{if } Y^h > L_2
\end{cases}
\quad \text{and} \quad
V^h = \begin{cases}
1 & \text{if } Y^h < L_2 \\
0 & \text{if } Y^h > L_2
\end{cases}.
\]
Motivation of our Penalty Function

Conjugate prior
(Hypothetical single proof load design)

- Proof load in Y mode.
- Break in X mode.

Hypothetical observations \((x^h_i, y^h_i)’s\).

Likelihood
(Single proof load design)

- Proof load in X mode.
- Break in Y mode.

Observations \((x_i, y_i)’s\).

\[
\begin{align*}
L_H(\theta; x^h, y^h) \quad &\quad \Downarrow \\
L(\theta; x, y) \quad &\quad \Downarrow \\
\end{align*}
\]

Posterior: 

\[
p(\theta|x, y, x^h, y^h) \propto L(\theta; x, y)L_H(\theta; x^h, y^h).
\]
Hyperparameters \( (x^h, y^h) \) specified by \( m \) independent prior hypothetical observations (Lele et al. (2006) and Novick et al. (1965)).

- Experts’ knowledge.
- Observations.
- Generator.
• Experts’ knowledge
  – The marginal parameters $(\mu_x^0, \sigma_x^0, \mu_y^0, \sigma_y^0)$.
  – sign of $\rho$.

• Random subsets
  – A random subset $\{x_1^*, x_2^*, \ldots, x_{k_h}^*\}$ of size $k_h$ from $\{x_1, \ldots, x_k\}$.
  – A random subset $\{y_{k_h+1}^*, y_{k_h+2}^*, \ldots, y_m^*\}$ of size $m - k_h$ from $\{y_1, \ldots, y_{n-k}\}$.

• Generator
  – $y_j^h = \sigma_y^0 \text{sign}(\rho)(x_j^* - \mu_x^0)/\sigma_x^0 + \mu_y^0$ for $j = 1, 2, \ldots, k_h$,
  – $x_j^h = \sigma_x^0 \text{sign}(\rho)(y_j^* - \mu_y^0)/\sigma_y^0 + \mu_x^0$ for $j = k_h + 1, \ldots, m$. 
10000 sample sets of size 300 from a bivariate Normal, $m_h = 30$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>CPSPLD Posterior mode</th>
<th>SPLD MLE</th>
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</thead>
<tbody>
<tr>
<td>$\mu_x = 1.846$</td>
<td></td>
<td></td>
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<tr>
<td>EST</td>
<td>1.846</td>
<td>1.846</td>
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<tr>
<td>RMSE</td>
<td>0.022</td>
<td>0.022</td>
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<tr>
<td>Variance</td>
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<td>Variance</td>
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<td>0.4483</td>
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Penalized Single Proof Load Design

The penalized maximum likelihood estimator $\hat{\theta}_p$ is defined as

$$\hat{\theta}_p = \arg\max_{\theta} \left[ \log L(\theta; x, y) + \lambda \log L_H(\theta; x^h, y^h) \right],$$

where $\lambda$ is a tuning parameter.

Selection of the tuning parameter:

- Cross-validation methods.
- Generalized information criterion.
Forest Products Stochastic Modeling Group

- Lumber data

- Forest Products Stochastic Modeling Group
4 Conclusion

Assessing relationship between destructive properties:

- Penalized Single Proof Load Design:
  
  - Penalty function motivated by a Bayesian conjugate prior.
  
  - Hyper-parameters estimated by empirical Bayes method.
  
  - Physically possible and satisfactory finite sample performance.
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Questions?